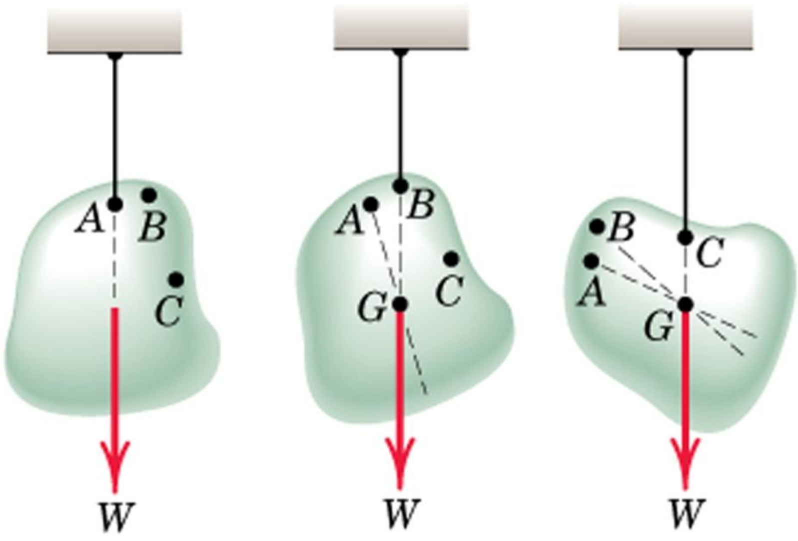
Center of Mass and Centroids

Center of Mass



A body of mass *m* in equilibrium under the action of tension in the cord, and resultant *W* of the gravitational forces acting on all particles of the body.
-The resultant is collinear with the cord

Suspend the body at different points

-Dotted lines show lines of action of the resultant force in each case.
-These lines of action will be concurrent at a single point G

As long as dimensions of the body are smaller compared with those of the earth.

- we assume uniform and parallel force field due to the gravitational attraction of the earth.

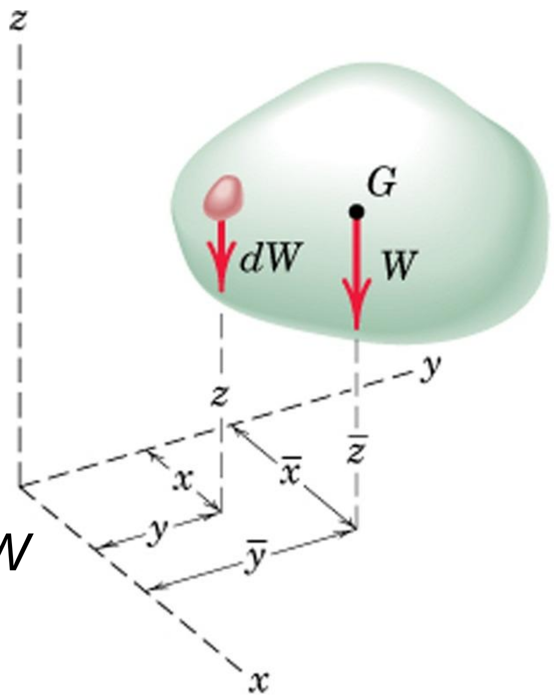
The unique **Point G** is called the Center of Gravity of the body (CG)

Center of Mass and Centroids

Determination of CG

- Apply Principle of Moments

Moment of resultant gravitational force W about any axis equals sum of the moments about the same axis of the gravitational forces dW acting on all particles treated as infinitesimal elements.



Weight of the body $W = \int dW$

Moment of weight of an element (dW) @ x-axis = ydW

Sum of moments for all elements of body = $\int ydW$

From Principle of Moments: $\int ydW = \bar{y}\, W$

$$\bar{x} = \frac{\int xdW}{W} \qquad \bar{y} = \frac{\int ydW}{W} \qquad \bar{z} = \frac{\int zdW}{W}$$

Moment of dW @ z axis???

= 0 or, ≠ 0

→ Numerator of these expressions represents the sum of the moments; Product of W and corresponding coordinate of G represents the moment of the sum → Moment Principle.

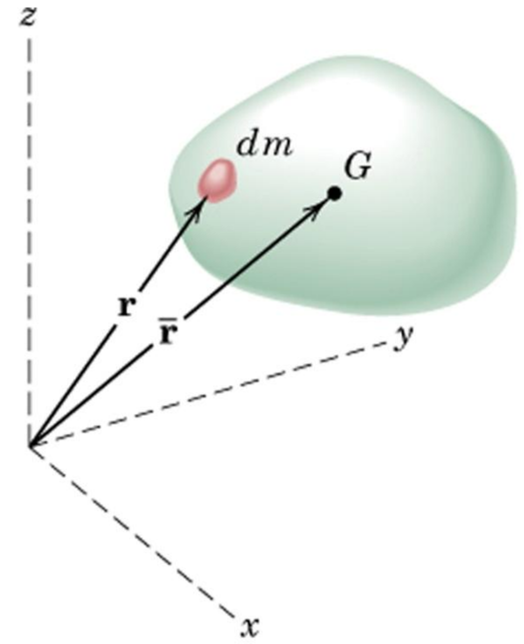
Center of Mass and Centroids

Determination of CG

$$\bar{x} = \frac{\int x dW}{W} \quad \bar{y} = \frac{\int y dW}{W} \quad \bar{z} = \frac{\int z dW}{W}$$

Substituting $W = mg$ and $dW = gdm$

→ $$\bar{x} = \frac{\int x dm}{m} \quad \bar{y} = \frac{\int y dm}{m} \quad \bar{z} = \frac{\int z dm}{m}$$



In vector notations:

Position vector for elemental mass: $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

Position vector for mass center G: $\bar{\mathbf{r}} = \bar{x}\mathbf{i} + \bar{y}\mathbf{j} + \bar{z}\mathbf{k}$

$$\bar{\mathbf{r}} = \frac{\int \mathbf{r} dm}{m}$$

Density ***ρ*** of a body = **mass per unit volume**

→Mass of a differential element of volume *dV* → $dm = \rho dV$

→ *ρ* may not be constant throughout the body

$$\bar{x} = \frac{\int x \rho dV}{\int \rho dV} \quad \bar{y} = \frac{\int y \rho dV}{\int \rho dV} \quad \bar{z} = \frac{\int z \rho dV}{\int \rho dV}$$

Center of Mass and Centroids

Center of Mass: Following equations independent of *g*

$$\bar{x} = \frac{\int x\,dm}{m} \qquad \bar{y} = \frac{\int y\,dm}{m} \qquad \bar{z} = \frac{\int z\,dm}{m}$$

$$\bar{\mathbf{r}} = \frac{\int \mathbf{r}\,dm}{m}$$ (Vector representation)

$$\bar{x} = \frac{\int x\rho\,dV}{\int \rho\,dV} \qquad \bar{y} = \frac{\int y\rho\,dV}{\int \rho\,dV} \qquad \bar{z} = \frac{\int z\rho\,dV}{\int \rho\,dV}$$

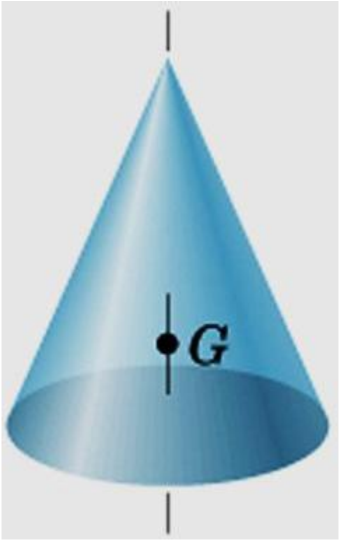
→**Unique point** [= $f(\rho)$] :: Centre of Mass (CM)

→CM coincides with CG as long as gravity field is treated as uniform and parallel

→CG or CM may lie outside the body

Center of Mass and Centroids

- **Symmetry**
  – *CM always lie on a **line** or a **plane of symmetry** in a homogeneous body*

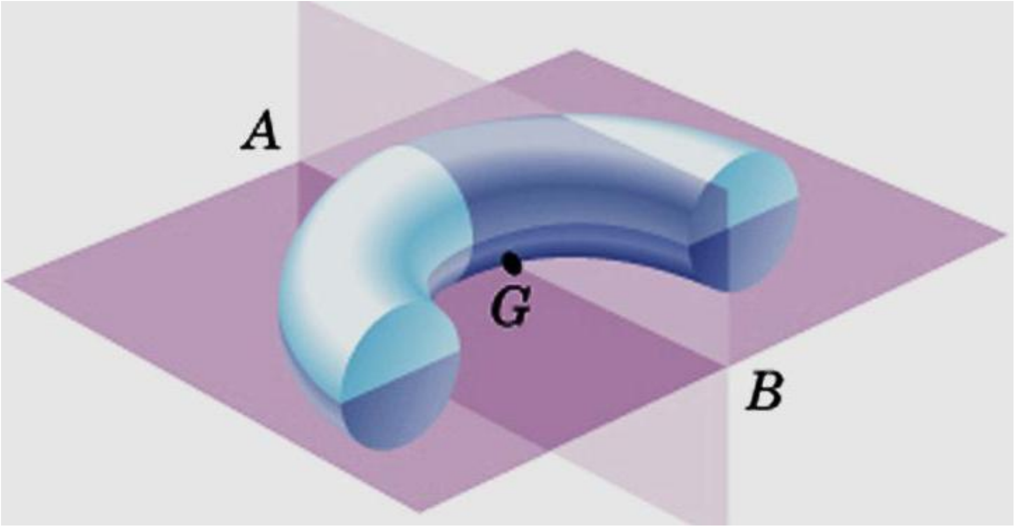


Right Circular Cone

CM on central axis

Half Right Circular Cone

CM on vertical plane of symmetry



Half Ring

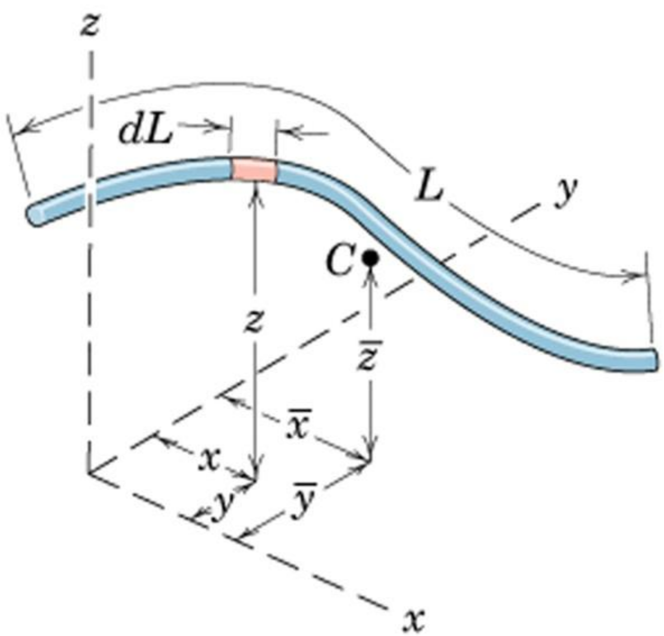
CM on intersection of two planes of symmetry (line AB)

Center of Mass and Centroids

Centroid

- Geometrical property of a body

- Body of uniform density :: Centroid and CM coincide



$$\bar{x} = \frac{\int x\,dm}{m} \qquad \bar{y} = \frac{\int y\,dm}{m} \qquad \bar{z} = \frac{\int z\,dm}{m}$$

Lines: Slender rod, Wire

Cross-sectional area = *A*

ρ and *A* are constant over *L*

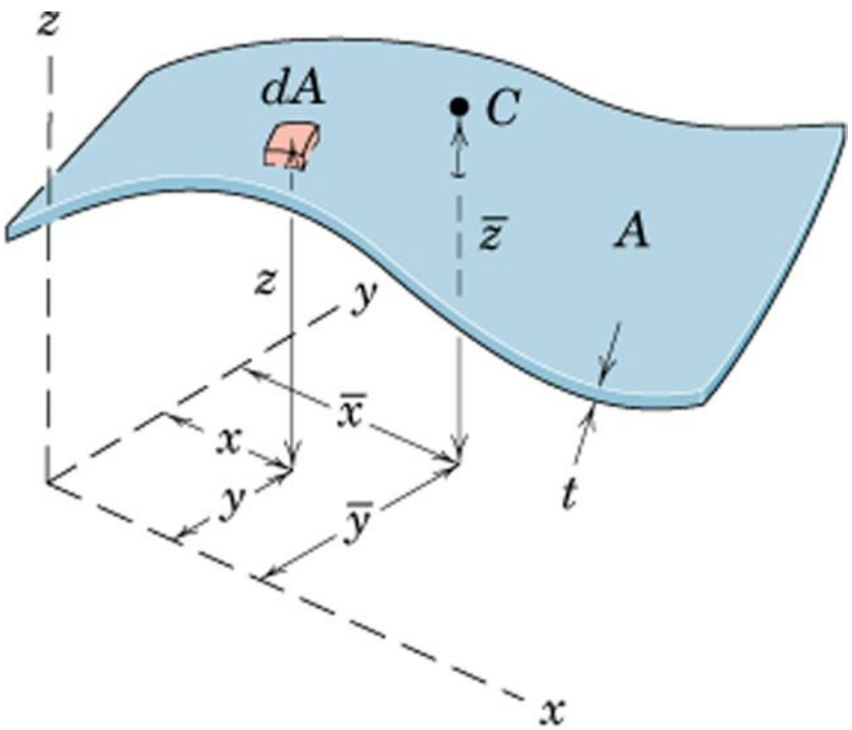
dm = *ρAdL*

Centroid and CM are the same points

$$\bar{x} = \frac{\int x\,dL}{L} \qquad \bar{y} = \frac{\int y\,dL}{L} \qquad \bar{z} = \frac{\int z\,dL}{L}$$

Center of Mass and Centroids

- Centroid



Areas: Body with small but constant thickness *t*

Cross-sectional area = *A*

ρ and *A* are constant over *A*

dm = *ρtdA*

Centroid and CM are the same points

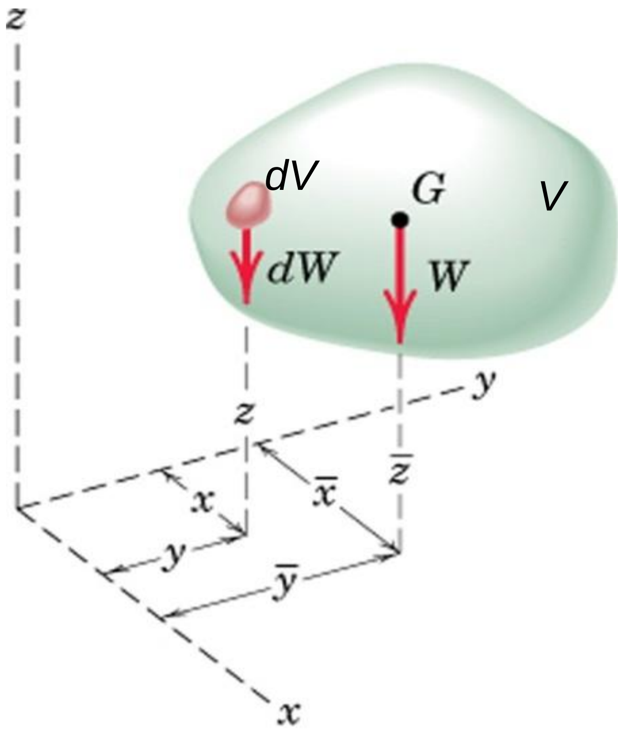
$$\bar{x} = \frac{\int x dm}{m} \quad \bar{y} = \frac{\int y dm}{m} \quad \bar{z} = \frac{\int z dm}{m}$$

$$\bar{x} = \frac{\int x dA}{A} \quad \bar{y} = \frac{\int y dA}{A} \quad \bar{z} = \frac{\int z dA}{A}$$

Numerator = First moments of Area

Center of Mass and Centroids

- Centroid



Volumes: Body with volume *V*

ρ constant over *V*

dm = ρdV

Centroid and CM are the same point

$$\bar{x} = \frac{\int x\,dm}{m} \qquad \bar{y} = \frac{\int y\,dm}{m} \qquad \bar{z} = \frac{\int z\,dm}{m}$$

$$\bar{x} = \frac{\int x\,dV}{V} \qquad \bar{y} = \frac{\int y\,dV}{V} \qquad \bar{z} = \frac{\int z\,dV}{V}$$

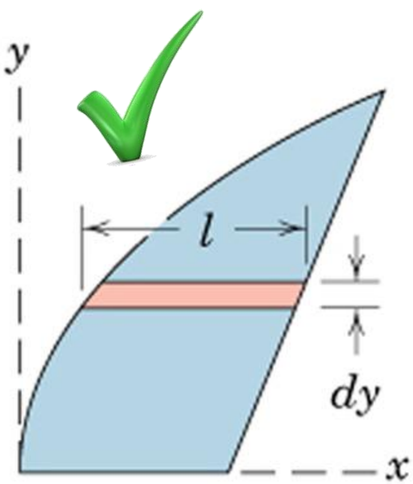
Numerator = First moments of Volume

Center of Mass and Centroid :: Guidelines

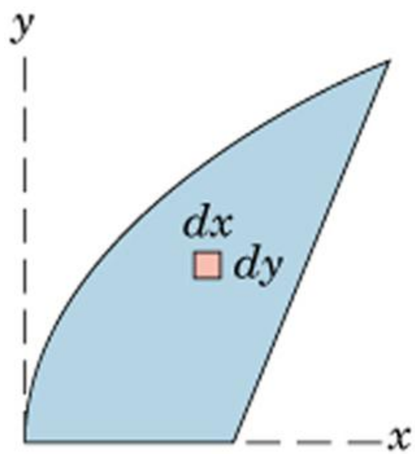
(a) Element Selection for Integration

- Order of Element

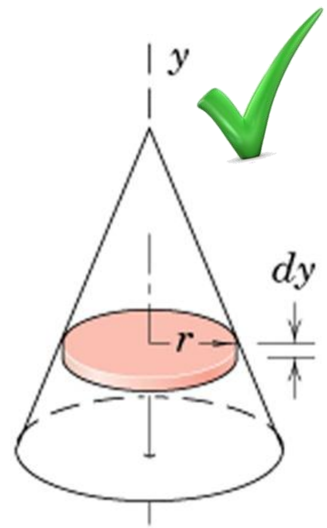
- First order differential element preferred over higher order element
- only one integration should cover the entire figure



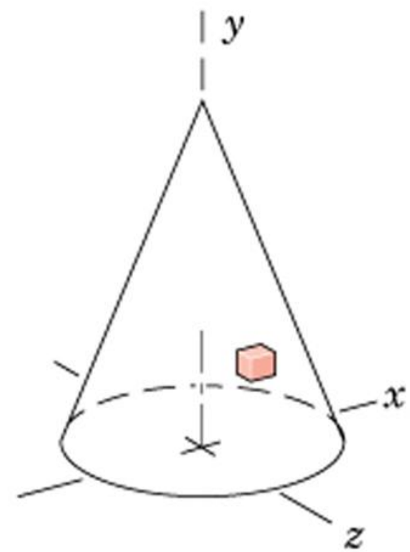
$A = \int dA = \int l\,dy$



$A = \int\int dx\,dy$



$V = \int dV = \int \pi r^2\,dy$



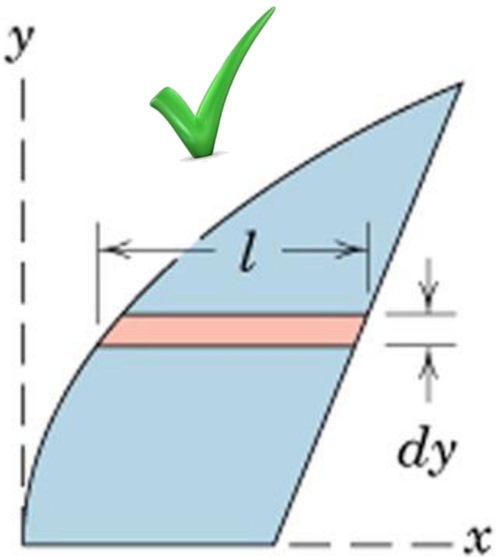
$V = \int\int\int dxdydz$

Center of Mass and Centroids :: Guidelines

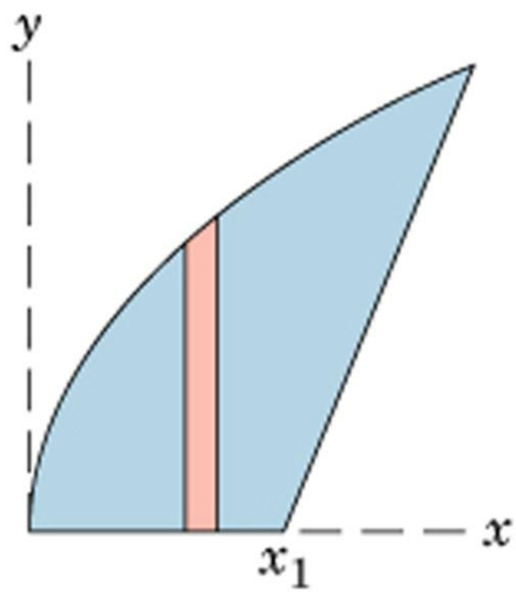
(b) Element Selection for Integration

- Continuity

- Integration of a single element over the entire area
- Continuous function over the entire area



Continuity in the expression for the width of the strip



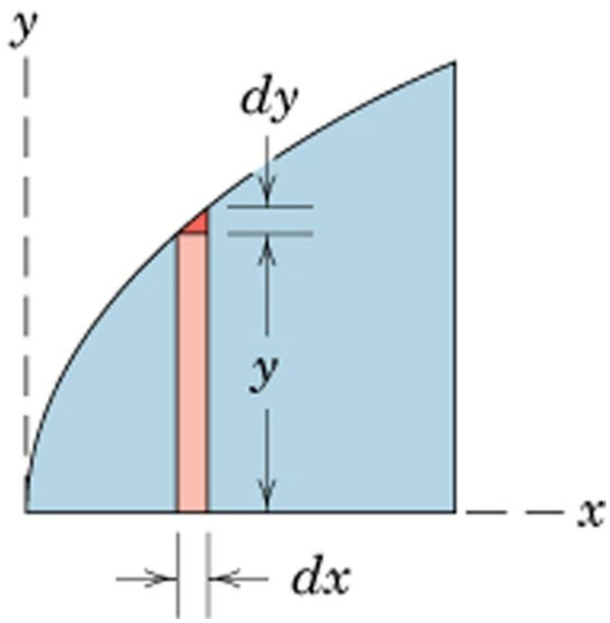
Discontinuity in the expression for the height of the strip at $x = x_1$

Center of Mass and Centroids :: Guidelines

(c) Element Selection for Integration

- Discarding higher order terms

- No error in limits



:: Vertical strip of area under the curve → $dA = ydx$

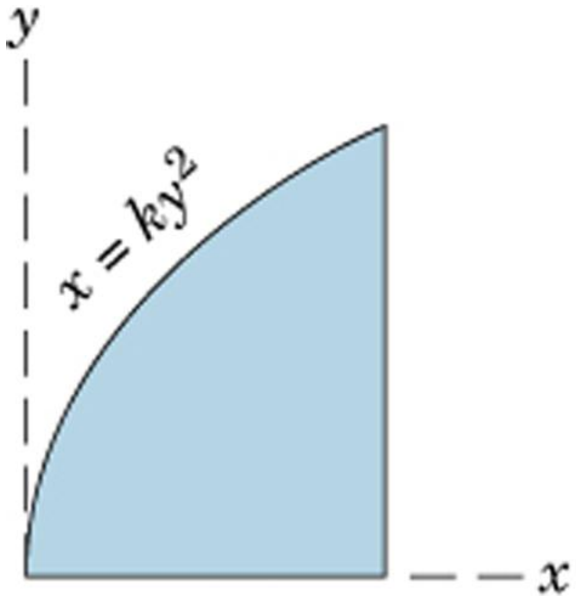
:: Ignore 2nd order triangular area $0.5dxdy$

Center of Mass and Centroids :: Guidelines

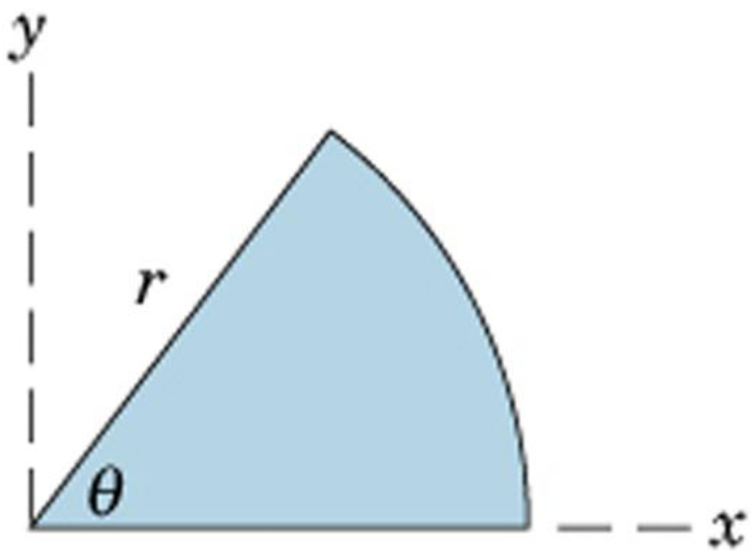
(d) Element Selection for Integration

- Coordinate system

- Convenient to match it with the



Curvilinear boundary
(Rectangular Coordinates)



Circular boundary
(Polar coordinates)

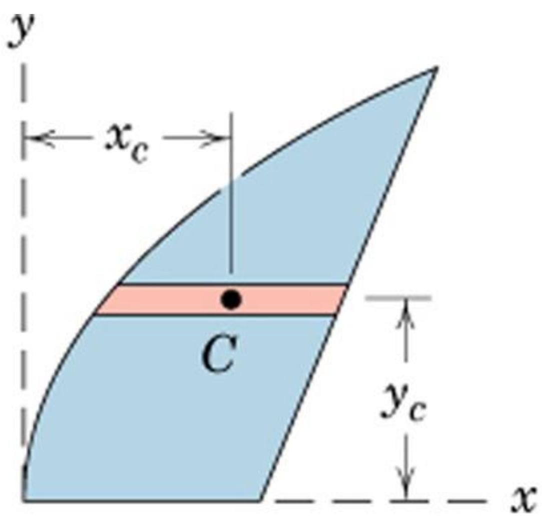
Center of Mass and Centroids :: Guidelines

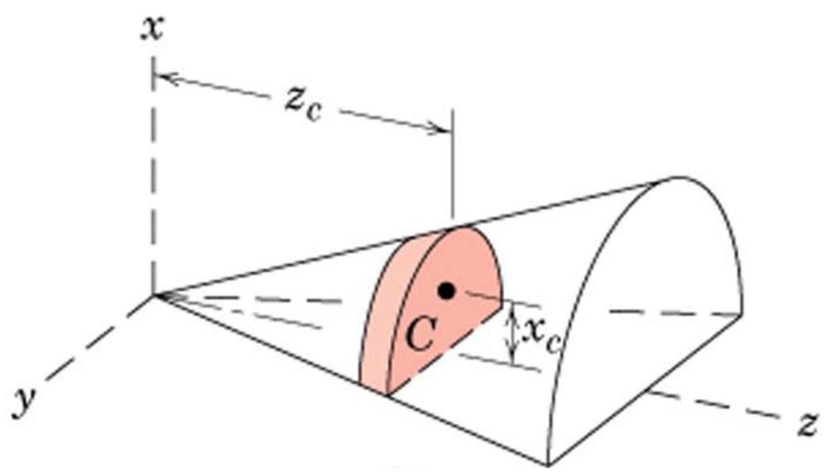
(e) Element Selection for Integration

- Centroidal coordinate (x_c, y_c, z_c) of element

- x_c, y_c, z_c to be considered for lever arm

:: not the coordinates of the area boundary





$$\bar{x} = \frac{\int x_c dV}{V} \quad \bar{y} = \frac{\int y_c dV}{V} \quad \bar{z} = \frac{\int z_c dV}{V}$$

Modified Equations

$$\bar{x} = \frac{\int x_c dA}{A} \quad \bar{y} = \frac{\int y_c dA}{A} \quad \bar{z} = \frac{\int z_c dA}{A}$$

Center of Mass and Centroids :: Guidelines

Centroids of Lines, Areas, and Volumes

1. Order of Element Selected for Integration
2. Continuity
3. Discarding Higher Order Terms
4. Choice of Coordinates
5. Centroidal Coordinate of Differential Elements

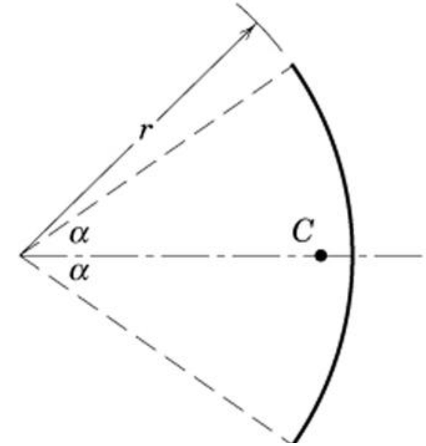
$$\bar{x} = \frac{\int x\,dL}{L} \qquad \bar{y} = \frac{\int y\,dL}{L} \qquad \bar{z} = \frac{\int z\,dL}{L}$$

$$\bar{x} = \frac{\int x_c\,dA}{A} \qquad \bar{y} = \frac{\int y_c\,dA}{A} \qquad \bar{z} = \frac{\int z_c\,dA}{A}$$

$$\bar{x} = \frac{\int x_c\,dV}{V} \qquad \bar{y} = \frac{\int y_c\,dV}{V} \qquad \bar{z} = \frac{\int z_c\,dV}{V}$$

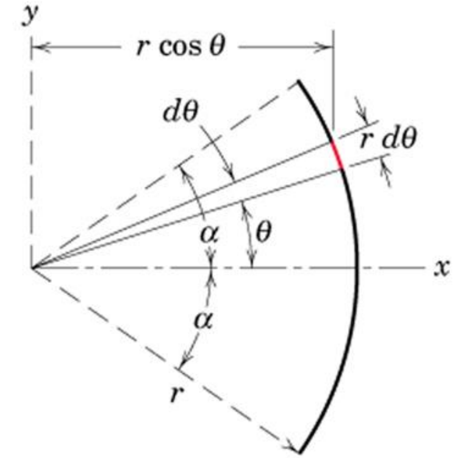
Example on Centroid :: Circular Arc

Locate the centroid of the circular arc



Solution: Polar coordinate system is better
Since the figure is symmetric: centroid lies on the x axis

Differential element of arc has length *dL = rdΘ*
Total length of arc*: L = 2αr*
x-coordinate of the centroid of differential element: *x=rcosΘ*

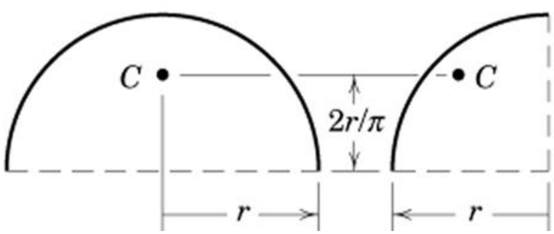


$$\bar{x} = \frac{\int x dL}{L} \quad \bar{y} = \frac{\int y dL}{L} \quad \bar{z} = \frac{\int z dL}{L}$$

$$[L\bar{x} = \int x \, dL] \qquad (2\alpha r)\bar{x} = \int_{-\alpha}^{\alpha} (r \cos\theta)\, r \, d\theta$$

$$2\alpha r\bar{x} = 2r^2 \sin\alpha$$

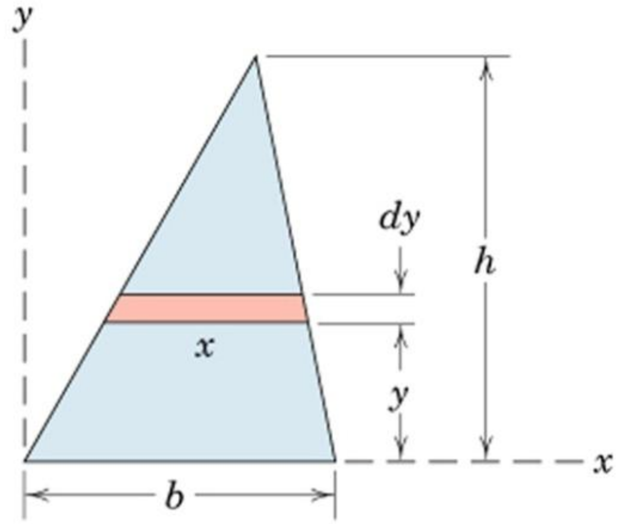
$$\bar{x} = \frac{r \sin\alpha}{\alpha}$$



For a semi-circular arc: *2α = π* → centroid lies at 2*r*/*π*

Example on Centroid :: Triangle

Locate the centroid of the triangle along h from the base



Solution:

$dA = x dy$; $x/(h-y) = b/h$

Total Area, A = ½(bh)

$$\bar{x} = \frac{\int x_c dA}{A} \quad \bar{y} = \frac{\int y_c dA}{A} \quad \bar{z} = \frac{\int z_c dA}{A}$$

$y_c = y$

$$\left[A\bar{y} = \int y_c \, dA\right] \qquad \frac{bh}{2}\bar{y} = \int_0^h y \frac{b(h-y)}{h} dy = \frac{bh^2}{6}$$

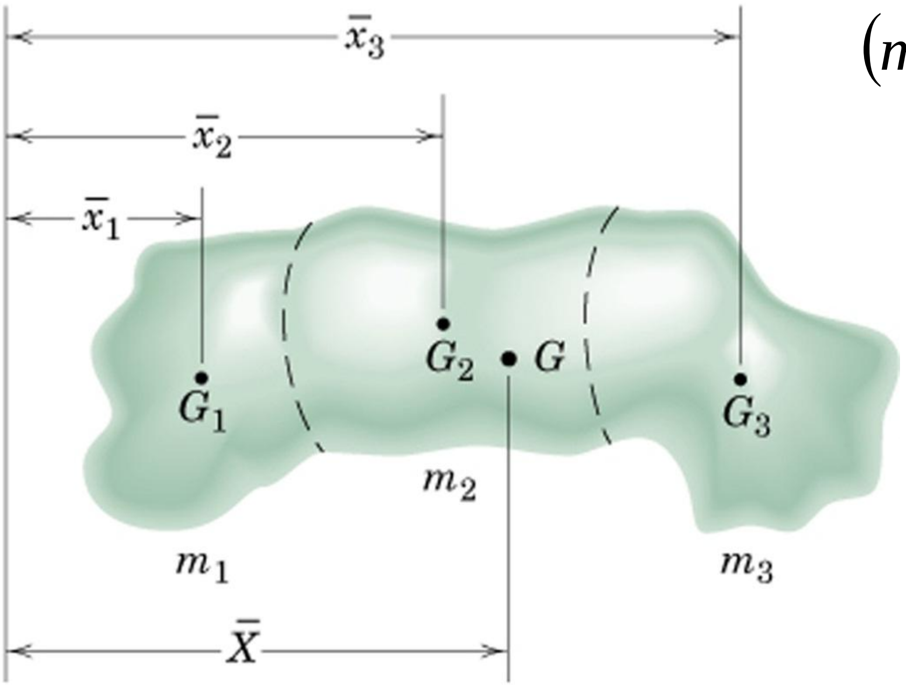
and $$\bar{y} = \frac{h}{3}$$

Center of Mass and Centroids

Composite Bodies and Figures

Divide bodies or figures into several parts such that their mass centers can be conveniently determined

→ Use Principle of Moment for all finite elements of the body



x-coordinate of the center of mass of the whole

$$(m_1 + m_2 + m_3)\bar{X} = m_1\bar{x}_1 + m_2\bar{x}_2 + m_3\bar{x}_3$$

Mass Center Coordinates can be written as:

$$\bar{X} = \frac{\sum m\bar{x}}{\sum m} \quad \bar{Y} = \frac{\sum m\bar{y}}{\sum m} \quad \bar{Z} = \frac{\sum m\bar{z}}{\sum m}$$

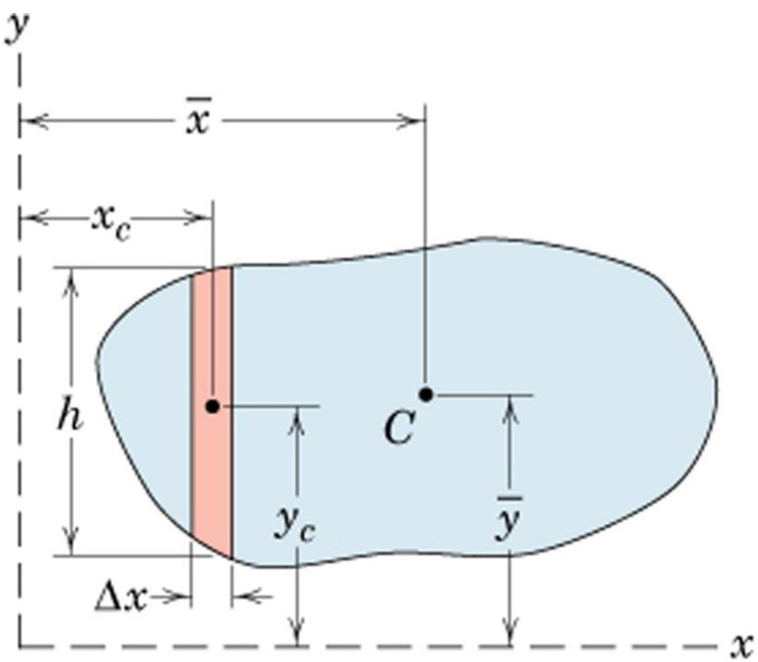
m's can be replaced by *L*'s, *A*'s, and *V*'s for lines, areas, and volumes

Centroid of Composite Body/Figure

Irregular area :: **Integration** vs **Approximate Summation**

- Area/volume boundary cannot be expressed analytically

- **Approximate summation** instead of integration



Divide the area into several strips
Area of each strip = $h\Delta x$
Moment of this area about x- and y-axis
= $(h\Delta x)y_c$ *and* $(h\Delta x)x_c$
→Sum of moments for all strips
divided by the total area will give
corresponding coordinate of the centroid

$$\bar{x} = \frac{\sum A x_c}{\sum A} \qquad \bar{y} = \frac{\sum A y_c}{\sum A}$$

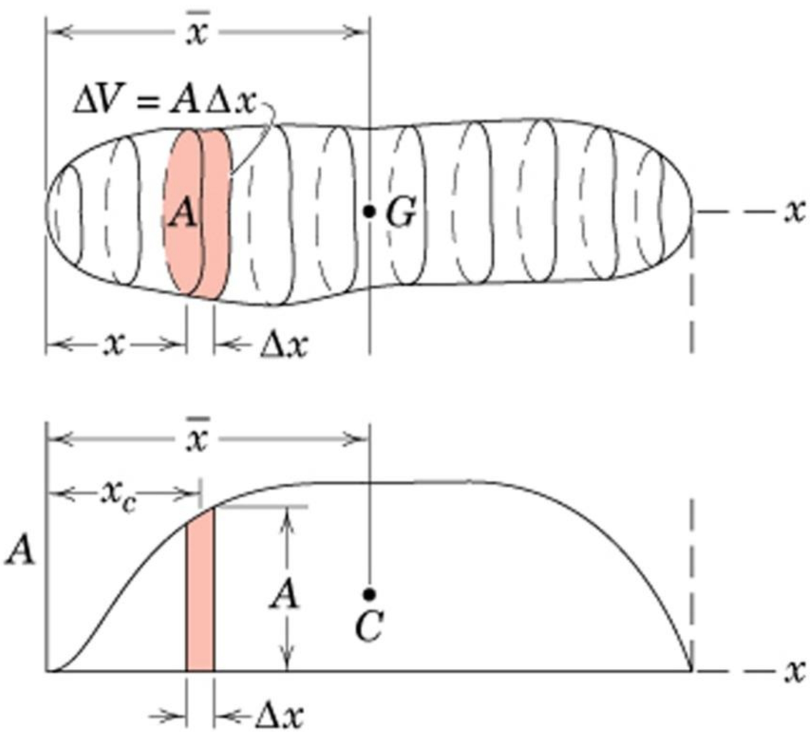
Accuracy may be improved by reducing the thickness of the strip

Centroid of Composite Body/Figure

Irregular volume :: **Integration** vs **Approximate Summation**

- Reduce the problem to one of locating the centroid of area

- **Approximate summation** instead of integration



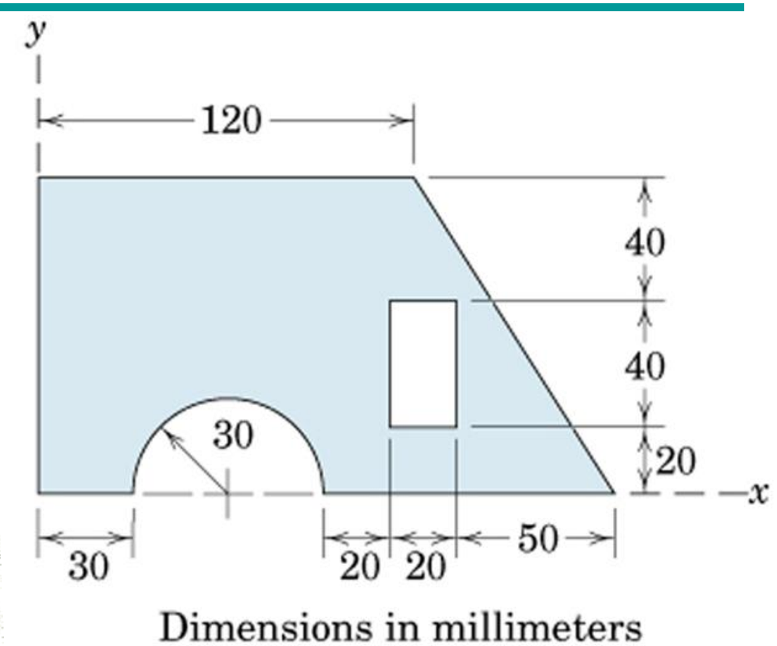
Divide the area into several strips
Volume of each strip = $A\Delta x$
Plot all such A against x.
→ Area under the plotted curve represents volume of whole body and the x-coordinate of the centroid of the area under the curve is given by:

$$\bar{x} = \frac{\sum (A\Delta x) x_c}{\sum A\Delta x} \Rightarrow \bar{x} = \frac{\sum V x_c}{\sum V}$$

Accuracy may be improved by reducing the width of the strip

Example on Centroid of Composite Figure

Locate the centroid of the shaded area

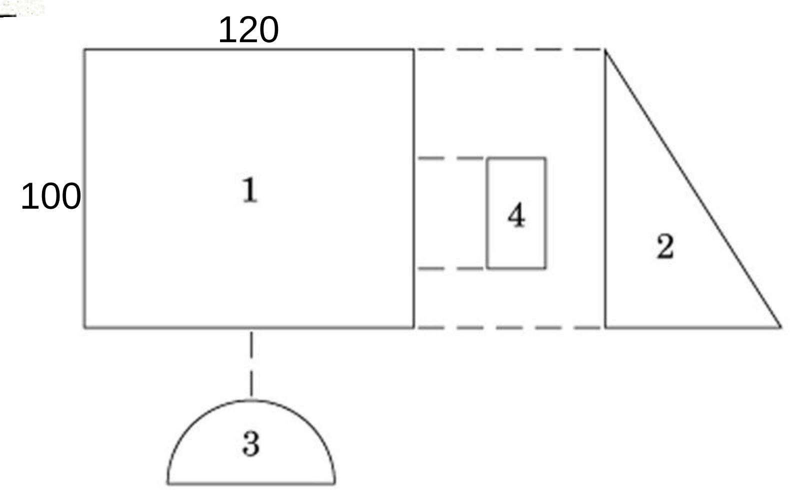


Solution: Divide the area into four elementary shapes: Total Area = $A_1 + A_2 - A_3 - A_4$

PART	A mm^2	$\bar{x}$ mm	$\bar{y}$ mm	$\bar{x}A$ mm^3	$\bar{y}A$ mm^3
1	12 000	60	50	720 000	600 000
2	3000	140	100/3	420 000	100 000
3	−1414	60	12.73	−84 800	−18 000
4	−800	120	40	−96 000	−32 000
TOTALS	12 790			959 000	650 000

$$\left[\bar{X} = \frac{\Sigma A\bar{x}}{\Sigma A}\right] \qquad \bar{X} = \frac{959\ 000}{12\ 790} = 75.0 \text{ mm}$$

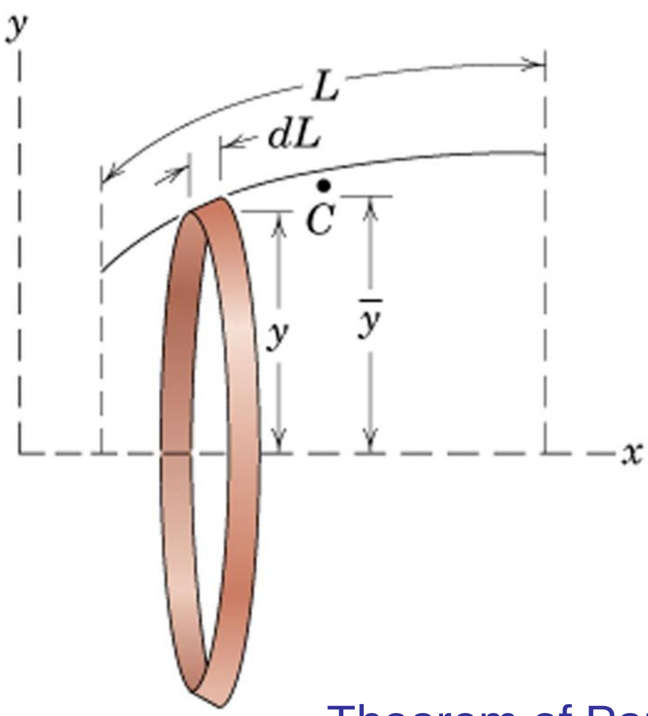
$$\left[\bar{Y} = \frac{\Sigma A\bar{y}}{\Sigma A}\right] \qquad \bar{Y} = \frac{650\ 000}{12\ 790} = 50.8 \text{ mm}$$



Center of Mass and Centroids

Theorem of Pappus: Area of Revolution

- method for calculating surface area generated by revolving a plane curve about a non-intersecting axis in the plane of the curve



Surface Area

Area of the ring element: circumference times *dL*

$dA = 2\pi y\, dL$

Total area, $A = 2\pi \int y\, dL$

$\because \bar{y}L = \int y\, dL \rightarrow \boxed{A = 2\pi \bar{y} L}$

If area is revolved through an angle $\theta < 2\pi$

θ in radians

or $\boxed{A = \theta \bar{y} L}$

$\bar{y}$ is the y-coordinate of the centroid C for the line of length *L*

Generated area is the same as the lateral area of a right circular cylinder of length L and radius $\bar{y}$

Theorem of Pappus can also be used to determine centroid of plane curves if area created by revolving these figures @ a non-intersecting axis is known