CE 601: Numerical Methods
Lecture 3

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System of Linear Algebraic Equations

Q. What are the ways to solve the system of linear equations? What are the types of solutions expected from a linear system?

E.g. consider a linear system:

\[
\begin{align*}
\begin{bmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
&=
\begin{bmatrix}
b_1 \\
b_2
\end{bmatrix}
\end{align*}
\]
• The above system represents two straight lines. Therefore we can have following types of solutions.

- Unique solution: One point where the lines intersect.
- No solution: Lines are parallel.
- Infinite solutions: Lines are the same line.
- Homogeneous: Solution is zero, trivial case.
• Q. How do you solve such system of linear equations?
• There are two approaches:
  o Direct elimination methods
  o Iterative methods
• **The Direct Elimination:**

• As the name suggests the methods are having procedures of algebraic elimination of the contents in the coefficient matrix that lead to solution.
  
  o Gauss elimination
  
  o Gauss-Jordan
  
  o Matrix inverse
  
  o LU factorization etc.
• In iterative methods, initially a solution is assumed and through iterations the actual solution is approached asymptotically.
  o Jacobi iteration
  o Gauss-Seidel iteration
  o Successive over relaxation
• **Matrix Properties:**
• We have seen earlier the system of linear equations can be represented by matrix methods.
• Q. What is a matrix?
• It is an array of elements that are arranged in orderly rows and columns.

\[
[A] = \begin{bmatrix}
 a_{ij}
\end{bmatrix}_{n \times m}
= \begin{pmatrix}
 a_{11} & \cdots & a_{1m} \\
 \vdots & \ddots & \vdots \\
 a_{n1} & \cdots & a_{nm}
\end{pmatrix}
\]

• Vectors: Column vector \( x = x_i = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \)

Row vector \( y = [y_j] = y_1 \ y_2 \ \cdots \ y_m \)
• Unit vector -> The vector whose magnitude is 1.

\[ \hat{i} = i = \begin{bmatrix} i_1 \\ i_2 \\ \vdots \\ i_n \end{bmatrix} \quad \text{and} \quad \sqrt{i_1^2 + i_2^2 + \ldots + i_n^2} = 1 \]

• You know what is meant by
  o Square matrix
  o Diagonal matrix
  o Identity matrix
  o Triangular matrix: 1) Upper and 2) Lower
• Also recall that: Matrix addition and Matrix multiplication.

• As a reading exercise please find the properties of matrix: 1. Associative, 2. Commutative and 3. Distributive.

• Square matrices -> properties
• You have if \([A]\) is a \(n \times n\) matrix, then \([A][A]^{-1} = I\)
  or, \([A]^{-1}[A] = I\)
  if there are two matrices \([A]\) and \([B]\) such that \([A][B] = I\) then \([A] = [I][B]^{-1}\)
• **Matrix Factorization**
• A matrix can be represented as product of two other matrices \([A] = [B][C]\)
• For a system of linear algebraic equations

\[ A \ x = b \]

\[ \sum_{j=1}^{n} a_{i,j} x_i = b_i; \quad i = 1, 2, 3, \ldots, n. \]

• We can do three row operations on such a linear system that will not alter the solution
  o Scaling
  o Pivoting
  o Elimination

• These row operations are extensively used in eliminations methods.
• **Direct Elimination Method**

• To perform elimination methods to find the solution of linear algebraic system we need to do row operations.

\[
\begin{pmatrix}
\begin{array}{ccc}
 a_{11} & \cdots & a_{1n} \\
 \vdots & \ddots & \vdots \\
 a_{n1} & \cdots & a_{nn}
\end{array}
\end{pmatrix}
\begin{pmatrix}
 x_1 \\
 \vdots \\
 x_n
\end{pmatrix} =
\begin{pmatrix}
 b_1 \\
 \vdots \\
 b_n
\end{pmatrix}
\]

i.e., \( A \ x = b \)
• Scaling: Any row can be multiplied by a constant. This is not going to change the solution.

• Pivoting: We can interchange the order of rows as per our convenience.

• Elimination: We can replace any row (i.e. a equation) by a weighted linear combination of that row with another row. This may yield some zeroes in that row. This is elimination.

• The row operation are not going to change the solutions.
• Q. So why do we require to do row operations?
• To prevent division by zero.
• To avoid round-off.
• To implement systematic elimination.
• Consider the following linear system example:

\[
\begin{pmatrix}
80 & -20 & -20 \\
-20 & 40 & -20 \\
-20 & -20 & 130
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix}
=
\begin{pmatrix}
20 \\
20 \\
20
\end{pmatrix}
\]
• How do you use to solve such a system:

\[
\begin{bmatrix}
80 & -20 & -20 & 20 \\
-20 & 40 & -20 & 20 \\
-20 & -20 & 130 & 20
\end{bmatrix}
\]

\[
R_2 = R_1 + 4Q_{21}
\]

\[
R_3 = R_1 + 4Q_{31}
\]

\[
\begin{bmatrix}
80 & -20 & -20 & 20 \\
0 & 140 & -100 & 100 \\
0 & -100 & 500 & 100
\end{bmatrix}
\rightarrow
\begin{bmatrix}
80 & -20 & -20 & 20 \\
0 & 140 & -100 & 100 \\
0 & 0 & 3000 & 1200
\end{bmatrix}
\]

Now \[3000 \times 3 = 1200\]

\[\therefore x_2 = 0.40\]
• Back substituting,
  
  \[140x_2 - 100x_3 = 100\]

  \[\Rightarrow x_2 = 1.0\]

  \[80x_1 - 20 \times 1.0 - 20 \times 0.40 = 20\]

  \[\Rightarrow x_1 = 0.60\]

  \[\Rightarrow \text{this is a simple elimination method.}\]

  \[\Rightarrow \text{In this process you were actually performing some row operations. You were not knowing them in school days.}\]
• Q. Why do you require scaling?
• As seen in example, we were able to multiply some rows with scalar values. This helped in subsequent elimination
• Q. Why do you require pivoting?
• In such linear systems the elements in major diagonal of the matrix is given $a_{ii}$ where $i = 1, 2, 3, ..., n$.
• If any $a_{ii} = 0$, then you will face difficulty in the above simple elimination method.
• To avoid that we can do
  o Interchanging of rows (equations)
  o Interchanging of columns (variables)

• This is called pivoting.

• If both rows and columns interchanged, it’s full pivoting else partial pivoting.

• Advantage:
  o We can avoid zero point elements
  o Reduce round-off errors
• Consider the system

\[
\begin{pmatrix}
0 & 2 & 1 \\
4 & 1 & -1 \\
-2 & 3 & -3
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix} =
\begin{pmatrix}
5 \\
-3 \\
5
\end{pmatrix}
\]

• We can \( a_{11} = 0 \), the largest element in first column is in row 2. Interchange row 1 and row 2

\[
\begin{pmatrix}
4 & 1 & -1: -3 \\
0 & 2 & 1 : 5 \\
-2 & 3 & -3 : 5
\end{pmatrix}
\]

\( R_3 = R_1 + 2R_3 \)

\[
\Rightarrow
\begin{pmatrix}
4 & 1 & -1: -3 \\
0 & 2 & 1 : 5 \\
0 & 7 & -7 : 7
\end{pmatrix}
\]

• By scaling we can reduce round-off errors.
Gauss Elimination Method

• To solve a linear system \([A]{x}={b}\), we have to do row operations:
  
  o Scaling  
  o Pivoting  
  o Elimination  

• While discussing about scaling we saw the example problem.

\[
\begin{pmatrix}
3 & 2 & 105 \\
2 & -3 & 103 \\
1 & 1 & 3
\end{pmatrix}
\begin{pmatrix}
{x_1} \\
{x_2} \\
{x_3}
\end{pmatrix} =
\begin{pmatrix}
104 \\
98 \\
3
\end{pmatrix}
\]
• If the computer program has restriction of three significant digits, then we saw that if we do direct elimination, we are getting erroneous results $x_1 = -0.844$, $x_2 = 0.924$, and $x_3 = 0.997$.

• The errors can be reduced by first doing scaling on the equation and determine its position in the system (i.e. pivoting).

• In direct elimination, we wanted the elements below pivot element as zero
  ✓ If we scale the numbers, we can pivot the appropriate equation.
✓ e.g. for $a_{11} = 3$, we want $a_{21} = a_{31} = 0$.
✓ Check the relative values.
   First column values $3 \ 2 \ 1^T$
   w.r.t. the largest values in their equation.

$$\begin{align*}
&\begin{bmatrix}
3/105 \\
2/103 \\
1/3
\end{bmatrix} = \\
&\begin{bmatrix}
0.0288 \\
0.0194 \\
0.333
\end{bmatrix}
\end{align*}$$

✓ This shows that the last row is having the largest scaled values. Therefore it will be appropriate if we pivot this element.
• Therefore pivoting is done by interchanging Row 1 and Row 3.

i.e.,

\[
\begin{pmatrix}
1 & 1 & 3 : 3 \\
2 & -3 & 103 : 98 \\
3 & 2 & 105 : 104
\end{pmatrix}
\]

• Now do row operations – elimination

\[
R_2 = R_2 - \left(\frac{a_{21}}{a_{11}}\right)R_1 = R_2 - 2R_1
\]
\[
R_3 = R_3 - \left(\frac{a_{31}}{a_{11}}\right)R_1 = R_3 - 3R_1
\]

\[
\begin{pmatrix}
1 & 1 & 3 : 3 \\
0 & -5 & 97 : 92 \\
0 & -1 & 96 : 95
\end{pmatrix}
\]
• Before doing second elimination, i.e., making $a_{32} = 0$, another round of scaling is done to determine pivoting.

\[
\begin{pmatrix} 1 & 1 & 3 \div 3 \\ 0 & -5 & 97 \div 92 \\ 0 & -1 & 96 \div 95 \end{pmatrix} \rightarrow R_3 = R_3 - (a_{32} / a_{22})R_2 = R_3 - 0.2R_2
\]

\[
\begin{pmatrix} 1 & 1 & 3 \div 3 \\ 0 & -5 & 97 \div 92 \\ 0 & 0 & 76.6 \div 76.6 \end{pmatrix}
\]

\[
\Rightarrow x_3 = 1.0, \ x_2 = -1.0, \ x_3 = 1.0
\]
• Gauss Elimination Method in a Nutshell

• You know that the method is used to solve a linear system

\[ A \begin{bmatrix} x \end{bmatrix} = \begin{bmatrix} b \end{bmatrix} \]

\( \begin{bmatrix} n \times n & n \times 1 & n \times 1 \end{bmatrix} \)

• Using systematic elimination the above system is converted to

\[ U \begin{bmatrix} x \end{bmatrix} = \begin{bmatrix} y \end{bmatrix} \]

\( \begin{bmatrix} n \times n & n \times 1 & n \times 1 \end{bmatrix} \)

• \([U]\) -> upper triangular matrix

\( -> \) using backsubstitution the solutions \( x_1, x_2, x_3 \) are found.
• **The Algorithm**

\[
\begin{pmatrix}
a_{11} & \cdots & a_{1n} \\
\vdots & \ddots & \vdots \\
a_{n1} & \cdots & a_{nn}
\end{pmatrix}
\begin{pmatrix}
x_1 \\
\vdots \\
x_n
\end{pmatrix}
= 
\begin{pmatrix}
b_1 \\
\vdots \\
b_n
\end{pmatrix}
\]

• **Step 1**

  o reduce the elements of first column to zero, except the pivot element
  o The pivot element is \( a_{11} \). If \( a_{11} = 0 \),
    
    do pivoting \( a_{r1} = \max_{2 \leq i \leq n} |a_{i1}| \)
  o Identify multiplication factor for each row.
• The multiplying factors are

\[ l_{21} = \frac{a_{21}}{a_{11}}, l_{31} = \frac{a_{31}}{a_{11}}, \ldots, l_{i1} = \frac{a_{i1}}{a_{11}}, \ldots l_{n1} = \frac{a_{n1}}{a_{11}} \]

• The first column of matrix \( A \) is except \( a_{11} \), all other quantities have to be zero.

• Now we need to multiply the first row by the multiplying factor \( (l_{i1}) \) and deduct it from the corresponding row \( (i) \).
  
  o Due to this changes occur in vector\( \{b\} \) also.
For our convenience the step number is given or bracketed superscript and we can decide this calculation as:

\[ a_{ij}^{(1)} = a_{ij} - l_{i1}a_{1j} \]

\[ b_i^{(1)} = b_i - l_{i1}b_1 \quad ;i = 2, 3, 4, \ldots, n \text{ and } j = 1, 2, 3, 4, \ldots, n \]

After the first steps you have:

\[
\begin{pmatrix}
11 & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\
0 & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\
0 & a_{32} & \cdots & a_{3j} & \cdots & a_{3n} \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & a_{n2} & \cdots & a_{nj} & \cdots & a_{nn}
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3 \\
\vdots \\
x_n
\end{pmatrix} = 
\begin{pmatrix}
b_1 \\
b_2^{(1)} \\
b_3^{(1)} \\
\vdots \\
b_n^{(1)}
\end{pmatrix}
\]
• Step 2
  o Adjust similar procedure as in step 1.
  o However now the pivot element is $a_{22}^{(1)}$.
  o We need to eliminate elements in the second column below $a_{22}^{(1)}$.
  o If $a_{22}^{(1)} = 0$, then do pivoting $a_{r2} = \max_{3 \leq i \leq n} |a_{i2}^{(1)}|$
  o Compute multiplying factors for each row below row 2.
    i.e., $l_{32} = a_{32}^{(1)}/a_{22}^{(1)}$, $l_{42} = a_{42}^{(1)}/a_{22}^{(1)}$, ... $l_{n2} = a_{n2}^{(1)}/a_{22}^{(1)}$
    i.e., in general $l_{i2} = a_{i2}^{(1)}/a_{22}^{(1)}$; $i = 3, 4, 5, ..., n$
○ Eliminate all elements below $a_{22}^{(1)}$ as zeroes.

For that do

\[
\begin{align*}
ad^{(2)}_{ij} &= a^{(1)}_{ij} - l_{i2}a_{2j}^{(1)} \\
bd^{(2)} &= b^{(1)}_i - l_{i2}b_{2j}^{(1)}
\end{align*}
\]

\(i = 3, 4, 5, \ldots, n\) and \(j = 2, 3, 4, 5, \ldots, n\)