LECTURE 9

24-JAN-2014

CONSERVATION EQUATIONS

We have seen yesterday in a multi-specie moving fluid continuum, the Eulerian approach to write the conservation of any extensive property G_α is given as:

$$\int_U \frac{\partial g_\alpha}{\partial t}\, dU + \int_S g_\alpha (\vec{V}_{G\alpha} \cdot \hat{n})\, dS = \int_U I_\alpha\, dU$$

or for any arbitrary volume U

$$\frac{\partial g_\alpha}{\partial t} + \nabla \cdot (g_\alpha \vec{V}_{G\alpha}) = I_\alpha \longrightarrow (1)$$

where g_α → the density of the extensive property G_α

$\vec{V}_{G\alpha}$ → the velocity of propogation of the extensive property G_α (i.e property G for a α-specie)

I_α → Production of the property G_α per unit volume per unit time.

This general conservation principle can be applied for developing conservation equations for

→ Mass

→ Momentum

→ Energy

} → The various extensive properties

1) Mass conservation of a specie α

Yesterday again we discussed on this aspect and formed the equation:

$$\frac{\partial \rho_\alpha}{\partial t} + \nabla \cdot (\rho_\alpha \vec{V}_\alpha) = I_\alpha \longrightarrow (2)$$

Again using $B_\alpha = \rho_\alpha$ and $\vec{V}_{G\alpha} = \vec{V}^*$ in equation (1) we can form ~~the~~ another material derivative for ρ_α such that

$$\frac{D^* \rho_\alpha}{Dt} = \frac{\partial \rho_\alpha}{\partial t} + (\vec{V}^* \cdot \nabla) \rho_\alpha$$

$$\text{or} \quad \frac{D^* \rho_\alpha}{Dt} = \frac{\partial \rho_\alpha}{\partial t} + \nabla \cdot (\rho_\alpha \vec{V}^*) - \rho_\alpha (\nabla \cdot \vec{V}^*) \hookrightarrow (3)$$

We also know diffusive velocity for specie α

$$\hat{\vec{V}}_\alpha = \vec{V}_\alpha - \vec{V}^*$$

and diffusive mass flux

$$\vec{J}_\alpha^* = \rho_\alpha \hat{\vec{V}}_\alpha = \rho_\alpha (\vec{V}_\alpha - \vec{V}^*) \longrightarrow (a)$$

(fourier based on mass average velocity)

Utilising the information in (3) and (a) in (2) we will get:

$$\frac{\partial \rho_\alpha}{\partial t} + \underbrace{\nabla \cdot (\rho_\alpha \vec{V}^*)} + \nabla \cdot \underbrace{\vec{J}^*_\alpha} = I_\alpha \quad \longrightarrow (4)$$

$\rho_\alpha \vec{V}^*$ = Advective or convective component of mass flux of α - species

$\vec{J}^*_\alpha$ = Diffusive or conductive mass flux of α - specie

2) Mass conservation of the entire fluid

In equation (1) put $\vec{V}_{G\alpha} = \vec{V}^*$ and $\rho_\alpha = \rho$ (the density of entire fluid system)

We will now get

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{V}^*) = 0} \quad \longrightarrow (5)$$

Q: Why $I_\alpha = 0$ in equation (5)?

When talking about the entire fluid and considering the mass average velocity $\vec{V}^*$, new mass cannot be created or destroyed in the system. The mass of some species may transform to some other specie, etc, but new mass cannot be created. $\therefore \; I_\alpha = 0$.

Here equation (5) may also be represented as

$$\frac{D\rho}{Dt} + \rho (\nabla \cdot \vec{V}^*) = 0$$

For homogeneous incompressible fluid, the average density ρ will not vary w.r.t space and time.

Then the equation for continuity will be

$$\boxed{\nabla \cdot \vec{V}^* = 0} \longrightarrow (6)$$

3) Conservation of Linear Momentum of a specie α

In a fluid continuum that is developed on mass (i.e. fluid consist of mass particles), the average mass density is ρ, average mass velocity is $\vec{V}^*$.

→ The linear momentum is an extensive property of mass and can now be given as

$$\int_U \rho \vec{V}^* dU$$

→ Based on this, let us now consider: G_α → linear momentum and density of extensive property g_α = Linear momentum of species α per unit volume of multi-specie fluid.

i.e. $g_\alpha = \rho_\alpha \vec{V}_\alpha$

Also let $\vec{V}_{G\alpha}$ = velocity of propagation of property for α-specie

= velocity of ~~momentum of~~ α-species momentum particles.

= $\vec{V}_{m\alpha}$

$I_\alpha = I_{m\alpha}$ → Rate of production of linear momentum of α-specie per unit volume per unit time.

∴ Equation (1) the conservation equation becomes:

$$\boxed{\frac{\partial}{\partial t}(\rho_\alpha \vec{V}_\alpha) + \nabla \cdot (\rho_\alpha \vec{V}_\alpha \vec{V}_{m\alpha}) = I_{m\alpha}} \rightarrow (7)$$

Now the product $\vec{V}_\alpha \vec{V}_{m\alpha}$ is a dyadic product

(Recall from vector mechanics, if there are two vectors $\vec{u} = u_1 \hat{i}_1 + u_2 \hat{i}_2 + u_3 \hat{i}_3$ and $\vec{v} = v_1 \hat{i}_1 + v_2 \hat{i}_2 + v_3 \hat{i}_3$

then you can have dyadic product

$$\vec{u}\vec{v} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \{v_1 \quad v_2 \quad v_3\} = \begin{pmatrix} u_1 v_1 & u_1 v_2 & u_1 v_3 \\ u_2 v_1 & u_2 v_2 & u_2 v_3 \\ u_3 v_1 & u_3 v_2 & u_3 v_3 \end{pmatrix}$$

This dyadic product of two vectors give a <u>second rank tensor</u>.)

Now let us define this dyadic product so:

$\bar{\bar{J}}_{m\alpha}$ → Flux of momentum of species α with respect to a fixed frame of reference.

$= \rho_\alpha \vec{V}_\alpha \vec{V}_{m\alpha}$ (Please Note that this will be a second-rank tensor)

Again w.r.t mass average velocity $\vec{V}^*$, if we can define a diffusive momentum velocity ~~and~~

$$\hat{\vec{V}}^*_{m\alpha} = \cancel{\vec{\vartheta}_{m\alpha} - \vec{V}_\alpha} \quad \vec{V}_{m\alpha} - \vec{V}^*$$

then :

we can also suggest a diffusive momentum flux

$$\overline{\overline{J}}^*_{m_\alpha} = \cancel{\rho_l \vec{V}_\alpha V} \; g_\alpha \hat{\vec{V}}^*_{m\alpha} = \rho_\alpha \vec{V}_\alpha \left(\vec{V}_{m\alpha} - \vec{V}^* \right)$$

Now

$$\overline{\overline{J}}_{m\alpha} = \rho_\alpha \vec{V}_\alpha \vec{V}_{m\alpha}$$

$$= \rho_\alpha \vec{V}_\alpha \vec{V}^* + \overline{\overline{J}}^*_{m\alpha}$$

∴ Equation (7) becomes:

$$\boxed{\frac{\partial}{\partial t}(\rho_\alpha \vec{V}_\alpha) + \nabla \cdot \left(\rho_\alpha \vec{V}_\alpha \vec{V}^* \right) + \nabla \cdot \overline{\overline{J}}^*_{m\alpha} = I_{m\alpha}} \quad \text{(8)}$$

Again you may see advective and diffusive components in the conservation equation.

4) <u>Conservation of Linear Momentum for entire liquid</u>

~~For~~ the entire fluid,

Mass average velocity $= \vec{V}^*$

Mass density $= \rho$

Linear momentum density . $g_\alpha = \rho \vec{V}^*$

Now G_α = linear momentum for entire fluid

$$= \int_V \rho \vec{V}^* dV$$

$\vec{V}_{G\alpha}$ = velocity of propagation of linear momentum in entire fluid

$$= \vec{V}_m$$

I_α → The rate of production of linear momentum in entire fluid per unit volume per unit time

= Resultant of all external forces acting on the volume.

- If $\vec{F}_\alpha$ = external force per unit mass of species α acting on this α-specie particle

Then $I_\alpha = \sum_{\alpha=1}^{N} \rho_\alpha \vec{F}_\alpha$

∴ The general conservation equation becomes

$$\boxed{\frac{\partial}{\partial t}(\rho \vec{V}^*) + \nabla \cdot (\rho \vec{V}^* \vec{V}_m) = \sum_{\alpha=1}^{N} \rho_\alpha \vec{F}_\alpha} \rightarrow (9)$$

Note: $\vec{V}^* \vec{V}_m$ → dyadic product

∴ Let us define

$\bar{\bar{J}}_m$ = Momentum flux w.r.t fixed frame of reference

$= \rho \vec{V}^* \vec{V}_m$

$\bar{\bar{J}}_m^*$ = diffusive momentum flux w.r.t mass average velocity

$= \rho \vec{V}^* (\vec{V}_m - \vec{V}^*)$

Then $\bar{\bar{J}}_m = \bar{\bar{J}}_m^* + \rho \vec{V}^* \vec{V}^*$

∴ Equation (9) becomes :

$$\boxed{\frac{\partial}{\partial t}(\rho \vec{V}^*) + \nabla\cdot(\rho \vec{V}^* \vec{V}^*) + \nabla\cdot \bar{\bar{J}}_m^* = \sum_{\alpha=1}^{\tilde{}} \rho_\alpha \vec{F}_\alpha} \longrightarrow (10)$$

Again you can see in equation (10) that while evaluating linear momentum conservation ~~yo~~ there are advective momentum fluxes and conductive momentum fluxes.

Q: What is this conductive or diffusive momentum flux ?

$\bar{\bar{J}}_m^*$

→ This is actually nothing but the negative of the stress tensor

Recall in index notation ; stress tensor was given as :

$$\sigma_{ij} = -p\,\delta_{ij} + \tau_{ij}$$

$$\therefore \quad \bar{\bar{J}}_m^* = -\bar{\bar{\sigma}}$$

$$\text{or} \quad (J_m^*)_{ij} = -\sigma_{ij}$$

∴ The momentum flux tensor $\bar{\bar{J}}_m = \rho \vec{V}^* \vec{V}_m$

will now become

$$\bar{\bar{J}}_m = \rho \vec{V}^* \vec{V}^* + p\bar{\bar{\delta}} - \bar{\bar{\tau}}$$

$$\text{or} \quad \left((J_m)_{ij} = \rho V_i^* V_j^* + p\,\delta_{ij} - \tau_{ij} \quad ; \quad \begin{matrix} i = 1, 2, 3 \\ j = 1, 2, 3 \end{matrix} \right)$$