LECTURE 8
23-JAN-2014

The Reynold's Transport Theorem

We can also arrive at the Reynold's Transport theorem by utilising the material derivatives for volumes.

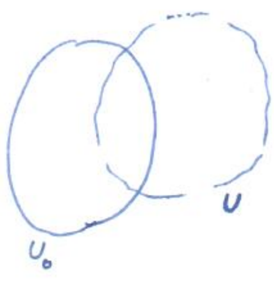
Recall that any extensive property G for an α-specie in a fluid continuum can also be expressed ~~as~~ in terms of its density g_α as:

$$G_\alpha(t) = \int_{[U(t)]} g_\alpha(\vec{x}, t)\, dU$$

where $G_\alpha(t) \rightarrow$ The amount of G_α instantaneously present in the volume $U(t)$

See that volume of continuum is moving and therefore it deforms (fluid).

i.e. Volume is function of time t. $g_\alpha(\vec{x}, t)$ is density of the extensive property at any spatial location in the continuum.



Again, recalling for a fluid mass particle - if its initial volume is given as dU_0 and represented by ~~position~~ material coordinates $\vec{X} = X_1 \hat{I}_1 + X_2 \hat{I}_2 + X_3 \hat{I}_3$.

When the fluid ~~moves~~ particle moves, we represent it through spatial coordinates $\vec{x}$.

where now $\vec{x} = \vec{x}(\vec{X})$ function of material coordinates.

In these 3-dimensional orthogonal Cartesian coordinate ~~system~~

let $dU_0 = dX_1 \, dX_2 \, dX_3$

This volume deforms to $dU = dx_1 \, dx_2 \, dx_3$

Aris (1962) suggested : $dU = J \, dU_0$

$J \rightarrow$ Jacobian $\rightarrow \frac{\partial x_i}{\partial X_j}$

From fluid mechanics we have seen that
Material derivative for this particle volume

$$\frac{D}{Dt}(dU) = \frac{D}{Dt}(J \, dU_0) = \frac{DJ}{Dt} \, dU_0$$

($\because$ dU_0 will not change with time)

$$\frac{DJ}{Dt} = (\nabla \cdot \vec{v}^*) J$$ (That is divergence of velocity vector).

$$\therefore \frac{D}{Dt}(dU) = \frac{DJ}{Dt} \, dU_0 = (\nabla \cdot \vec{v}^*) dU_0 J$$

1.1.b Now for the multi-specie component

$$\frac{DJ}{Dt} = (\nabla \cdot \vec{V}_{G\alpha}) J$$

$\Rightarrow$ $\therefore$ If we want to take material derivative of G_α

where $G_\alpha(t) = \int_{U(t)} g_\alpha(\vec{x}, t) \, dU$,

then we need to express $g_\alpha(\vec{x}, t)$ and dU in terms of material coordinates.

i.e. We know for the fluid continuum initially the volume is U_0 and represented by material coordinate $\vec{X_\alpha}$

i.e. Our spatial coordinate at any instant will be.

$$\vec{x} = \vec{x}\,(\vec{X_\alpha}, t)$$

$$\therefore \frac{D}{Dt} G_\alpha = \frac{D}{Dt}\left[\int_{U(t)} g_\alpha(\vec{x}, t)\, dU\right]$$

$$= \int_{U_0} \frac{Dg_\alpha}{Dt} J\, dU_0 + \int_{U_0} g_\alpha \frac{DJ}{Dt}\, dU_0$$

$$= \int_{U_0}\left[\frac{\partial g_\alpha}{\partial t} + (\vec{V_{G\alpha}} \cdot \nabla)\, g_\alpha\right] J\, dU_0 + \int_{U_0} g_\alpha (\nabla \cdot \vec{V_{G\alpha}})\, J\, dU_0$$

$$= \int_{U_0}\left[\frac{\partial g_\alpha}{\partial t} + \nabla\cdot\left(g_\alpha \vec{V_{G\alpha}}\right)\right] J\, dU_0$$

$$= \int_{U}\left[\frac{\partial g_\alpha}{\partial t} + \nabla \cdot g_\alpha \vec{V_{G\alpha}}\right] dU$$

Now $\displaystyle\int_U \nabla\cdot(g_\alpha \vec{V_{G\alpha}})\, dU = \int_S g_\alpha (\vec{V_{G\alpha}} \cdot \hat{n})\, dS$

(Using Gauss divergence theorem).

$$\therefore \frac{DG_\alpha}{Dt} = \int_U \frac{\partial g_\alpha}{\partial t}\, dU + \int_S g_\alpha (\vec{V_{G\alpha}} \cdot \hat{n})\, dS$$

Cases

If in a volume U, there are ~~no~~ production of property G_α within the volume at a rate say I_α per unit volume per unit time.
(e.g. due to chemical, internal processes, etc.)

Then

$$\frac{DG_\alpha}{Dt} = \int_U I_\alpha \, dU$$

$$\therefore \int_U I_\alpha \, dU = \int_U \frac{\partial g_\alpha}{\partial t} dU + \int_S g_\alpha (\vec{V}_{G\alpha} \cdot \hat{n}) dS$$

The Eulerian point of view is represented.

That is the rate of increase of G_α within the volume U is equal to the sum of the rate at which this property G_α crosses into (or out) through the surface and the rate at which this property is produced within U.

$$\therefore \int_U \frac{\partial g_\alpha}{\partial t} dU + \int_S g_\alpha (\vec{V}_{G\alpha} \cdot \hat{n}) dS = \int I_\alpha \, dU$$

i.e. $$\int_U \left[\frac{\partial g_\alpha}{\partial t} + \nabla \cdot (g_\alpha \vec{V}_{G\alpha}) - I_\alpha \right] dU = 0$$

As this volume U is arbitrary we need to have then

$$\boxed{\frac{\partial g_\alpha}{\partial t} + \nabla \cdot (g_\alpha \vec{V}_{G\alpha}) = I_\alpha}$$

This equation is the general conservation principle of a property G of the specie α.

Equation of Mass

1) Mass conservation of α- specie

In the general conservation principle

$$\frac{\partial g_\alpha}{\partial t} + \nabla \cdot (g_\alpha \vec{V}_{G\alpha}) = I_\alpha$$

put $g_\alpha = \rho_\alpha$, $\vec{V}_{G\alpha} = \vec{V}_\alpha$

$$\therefore \frac{\partial \rho_\alpha}{\partial t} + \nabla \cdot (\rho_\alpha \vec{V}_\alpha) = I_\alpha$$

where I_α will be now $[M L^{-3} T^{-1}]$

→ the rate at which mass of α- specie is produced per unit volume of the system (Maybe by chemical reactions, etc.).

∴ Equation of continuity is:

$$\frac{\partial \rho_\alpha}{\partial t} + \nabla \cdot (\rho_\alpha \vec{V}_\alpha) = I_\alpha$$

~~For~~ Note that, when the material derivative was discussed

$$\left\{ \frac{DB_\alpha}{Dt} = \frac{\partial B_\alpha}{\partial t} + (\vec{V}_{G\alpha} \cdot \nabla) B_\alpha = \frac{\partial B_\alpha}{\partial t} + \nabla \cdot (\vec{V}_{G\alpha} B_\alpha) - B_\alpha (\nabla \cdot \vec{V}_{G\alpha}) \right\}$$

$$\frac{D\rho_\alpha}{Dt} = \frac{\partial \rho_\alpha}{\partial t} + (\vec{V}_\alpha \cdot \nabla)\rho_\alpha$$

Using that, we have for

$$\frac{\partial \rho_\alpha}{\partial t} + \nabla \cdot (\rho_\alpha \vec{V}_\alpha) = I_\alpha$$ ~~can be~~

can be written as:

$$\frac{D\rho_\alpha}{Dt} + \rho_\alpha (\nabla \cdot \vec{V}_\alpha) = I_\alpha \longrightarrow (1)$$

Again if we wish to evaluate in:

$$\frac{DB_\alpha}{Dt} = \frac{\partial B_\alpha}{\partial t} + (\vec{V}_{G\alpha} \cdot \nabla) B_\alpha$$

Put $B_\alpha = \rho_\alpha$, $\vec{V}_{G\alpha} = \vec{V}^*$ (Mass Average velocity)

Then $$\frac{D^*\rho_\alpha}{Dt} = \frac{\partial \rho_\alpha}{\partial t} + (\vec{V}^* \cdot \nabla) \rho_\alpha \rightarrow (2)$$

We also have $$\frac{D\rho_\alpha}{Dt} + \rho_\alpha (\nabla \cdot \vec{V}_\alpha) = I_\alpha \rightarrow (1)$$

(Obtained earlier).

From (1) and (2), we can now write:

$$\frac{D^*\rho_\alpha}{Dt} - (\vec{V}^* \cdot \nabla)\rho_\alpha + (\vec{V}_\alpha \cdot \nabla)\rho_\alpha + \rho_\alpha (\nabla \cdot \vec{V}_\alpha) = I_\alpha$$

i.e. $$\frac{D^*\rho_\alpha}{Dt} + \left[(\vec{V}_\alpha - \vec{V}^*) \cdot \nabla\right]\rho_\alpha + \rho_\alpha (\nabla \cdot \vec{V}_\alpha) = I_\alpha \rightarrow (3)$$

$$\cancel{= \frac{\partial \rho_\alpha}{\partial t} + (\vec{V}^* \cdot \nabla)\rho_\alpha + (\vec{J}_\alpha^* \cdot \nabla)\rho_\alpha + \rho_\alpha (\nabla \cdot \vec{V}_\alpha) = I_\alpha}$$

Also note that

$$\frac{D^*\rho_\alpha}{Dt} = \frac{\partial \rho_\alpha}{\partial t} + \nabla \cdot (\rho_\alpha \vec{V}^*) - \rho_\alpha (\nabla \cdot \vec{V}^*)$$

and $\hat{\vec{V}}_\alpha^* = \vec{V}_\alpha - \vec{V}^*$ and $\vec{J}_\alpha^* = \rho_\alpha \hat{\vec{V}}_\alpha^* = \rho_\alpha (\vec{V}_\alpha - \vec{V}^*)$

$\therefore$ In equation (3), we have:

$$\frac{\partial \rho_\alpha}{\partial t} + \nabla \cdot (\rho_\alpha \vec{V}^*) - \rho_\alpha (\nabla \cdot \vec{V}^*) + (\hat{\vec{V}}_\alpha^* \cdot \nabla)\rho_\alpha + \rho_\alpha (\nabla \cdot \vec{V}_\alpha) = I_\alpha$$

ie. $$\frac{\partial \rho_\alpha}{\partial t} + \nabla\cdot(\rho_\alpha \vec{V}^*) + \rho_\alpha\left[\nabla\cdot(\vec{V}_\alpha - \vec{V}^*)\right] + (\hat{\vec{V}}_\alpha^*\cdot\nabla)\rho_\alpha = I_\alpha$$

ie. $$\frac{\partial \rho_\alpha}{\partial t} + \nabla\cdot(\rho_\alpha \vec{V}^*) + \rho_\alpha(\nabla\cdot \hat{\vec{V}}_\alpha^*) + (\hat{\vec{V}}_\alpha^*\cdot\nabla)\rho_\alpha = I_\alpha$$

or $$\boxed{\frac{\partial \rho_\alpha}{\partial t} + \nabla\cdot(\rho_\alpha \vec{V}^*) + \nabla\cdot(\vec{J}_\alpha^*) = I_\alpha} \rightarrow (4)$$

$\rho\vec{V}^*$ → Convective component of the mass flux of the α-specie based on mass average velocity $\vec{V}^*$

$\vec{J}_\alpha^*$ → Diffusive component of the mass flux of the α-specie

From the above equation, we can now identify that there can be

- ↳ Advective flux
- ↳ Diffusive flux

} for the α-specie while considering the mass average velocity $\vec{V}^*$.

2) Mass conservation for entire fluid

For entire fluid, you have mass density $= \rho$.
Mass average velocity $= \vec{V}^*$.

Then the conservation equation will be:

$\frac{\partial \rho}{\partial t}$ $g_\alpha = \rho$, $\vec{V}_{a\alpha} = \vec{V}^*$,

$I_\alpha = 0$ ($\because$ Because mass cannot be produced in the system)

$$\therefore \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{V}^*) = 0$$

Case (i) For incompressible, homogeneous liquid you have

$$\boxed{\nabla \cdot \vec{V}^* = 0}$$