FUNDAMENTAL TRANSPORT EQUATIONS (CONTD ...)

In the last class, we discussed on :

→ The diffusive velocities

→ The diffusive fluxes

→ The Eulerian and Lagrangian co-ordinates ($\vec{x}$ & $\vec{X}$)

* The Lagrangian approach $\vec{x} = \vec{x}(\vec{X}, t)$
* The Eulerian approach $\vec{X} = \vec{X}(\vec{x}, t)$

→ For a multi-specie liquid, if the mass particles of α-specie are labelled using $\vec{X}_\alpha$

i.e. $\vec{X}_\alpha = X_{1_\alpha} \hat{I}_1 + X_{2_\alpha} \hat{I}_2 + X_{3_\alpha} \hat{I}_3$.

then the instantaneous velocity $\vec{V}_\alpha$ is :

$$\vec{V}_\alpha (\vec{X}_\alpha, t) = \left. \frac{\partial \vec{x}}{\partial t} \right|_{\vec{X}_\alpha = \text{constant}}$$

(Rate of change of position $\vec{x}$ of the particle at $\vec{X}_\alpha$).

→ The <u>substantial derivative</u> is to indicate partial differentiation w.r.t. time of any fluid property that follows a particular particle of that property (Lagrangian approach)

Symbolically $\frac{D}{Dt}(\,)$.

→ If we consider a property B_α of a G_α particle,

↳ The Lagrangian formulation for following particles

* gives temporal rate of change of B_α

↳ The particle's position is $\vec{x}$ or x_i

→ The velocity of the particle is

$$\vec{V}_{G_\alpha} \quad \text{or} \quad \left(V_{G\alpha}\right)_i = \frac{\partial x_i}{\partial t}(\vec{X}_\alpha, t) = \frac{Dx_i}{Dt}$$

We saw that:

$$\frac{DB_\alpha}{Dt} = \frac{\partial B_\alpha}{\partial t} + \left(\vec{V}_{G_\alpha} \cdot \nabla\right) B_\alpha$$

Local Derivative   Convective Derivative

Note:→ 1) If B_α is the density of the fluid mass particle (say ρ)

Then we have:

$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + \left(\vec{V}^* \cdot \nabla\right)\rho$$

For multi-specie fluid. if ρ_α in the property B_α

$$\frac{D\rho_\alpha}{Dt} = \frac{\partial \rho_\alpha}{\partial t} + \left(\vec{V}_\alpha \cdot \nabla\right)\rho_\alpha$$

2) If $B_\alpha = \vec{V}^*$, then $\left(\vec{V}^* = V_1^* \hat{i}_1 + V_2^* \hat{i}_2 + V_3^* \hat{i}_3\right)$

$$\frac{D\vec{V}^*}{Dt} = \frac{\partial \vec{V}^*}{\partial t} + \left(\vec{V}^* \cdot \text{grad}\right)\vec{V}^*$$

i.e. $\vec{a} = \frac{\partial \vec{V}^*}{\partial t} + \left(\vec{V}^* \cdot \nabla\right)\vec{V}^* \Rightarrow \left(\begin{array}{c}\text{Index Notation} \\ a_i = \frac{\partial V_i^*}{\partial t} + V_j^* \frac{\partial V_i^*}{\partial x_j}\end{array}\right)$

where $\vec{a}$ = acceleration

The expanded form of $\vec{a}$ ~~will be~~ in three dimensional Cartesian coordinates $\vec{a} = a_1 \hat{i}_1 + a_2 \hat{i}_2 + a_3 \hat{i}_3$

$$a_1 = \frac{\partial V_1^*}{\partial t} + V_1^* \frac{\partial V_1^*}{\partial x_1} + V_2^* \frac{\partial V_1^*}{\partial x_2} + V_3^* \frac{\partial V_1^*}{\partial x_3}$$

⇒ The surface of a fluid continuum is usually considered as composed of same particles. Therefore you see interfaces between air and water, etc.

Let us describe this surface as:

$$F(x_1, x_2, x_3, t) = 0$$

If we take $B_\alpha = F$ (Please note this is not force)

$$\frac{DF}{Dt} = \frac{\partial F}{\partial t} + (\vec{V}^{*} \cdot \nabla) F = 0$$

(i.e Material derivative is zero for the surface).

⇒ Again the material derivative equation:

$$\frac{DB_\alpha}{Dt} = \frac{\partial B_\alpha}{\partial t} + \left((V_{G\alpha})_j \frac{\partial B_\alpha}{\partial x_j} \right)$$

This can also be written as:

$$\frac{DB_\alpha}{Dt} = \frac{\partial B_\alpha}{\partial t} + \frac{\partial}{\partial x_j} \left(V_{G\alpha_j} B_\alpha \right) - B_\alpha \frac{\partial V_{G\alpha_j}}{\partial x_j}$$

$$\left(\text{or} \quad \frac{DB_\alpha}{Dt} = \frac{\partial B_\alpha}{\partial t} + \nabla \cdot \left(\vec{V}_{G\alpha} B_\alpha \right) - B_\alpha \left(\nabla \cdot \vec{V}_{G\alpha} \right) \right)$$

For certain cases you have $\frac{\partial B_\alpha}{\partial t} + \nabla \cdot \left(\vec{V}_{G\alpha} B_\alpha \right) = 0$

Then you have: $\nabla \cdot \vec{V}_{G\alpha} = -\frac{1}{B_\alpha} \frac{DB_\alpha}{Dt}$

The Conservation Principle in General form

The derivation or developments of the general conservation principle might have already studied in your Advanced Fluid Mechanics and Advanced Hydrology courses. However, we may again see its derivation briefly suiting our needs.

⇒ Let us consider a multi-component flowing fluid occupying (say 3D Cartesian coordinate) space.

→ The entire fluid is continuum.

→ Now recall, when we discussed on particles, we can ~~form~~ suggest particles for each species

→ Therefore, we can now consider the entire fluid continuum consisting of many continua associated with particles of each species occupying the same space and time.

⇒ Now consider an initial amount of G_α of an extensive property (say mass, momentum, etc.) in the space and occupying a volume U and enclosed by surface S, at a particular instant t.

As the property G_α is moving, the volume of the continuum U continuously changes. i.e. $U = U(\vec{x}, t)$

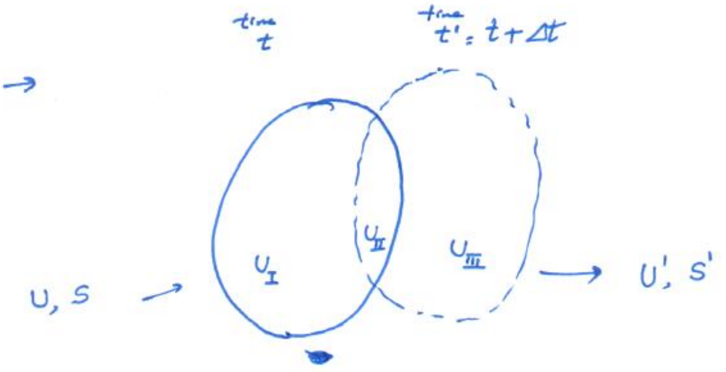
Now In Lagrangian approach you have $\vec{x} = \vec{x}(\vec{X}, t)$

The time rate of change of G_α in Lagrangian approach is $\left.\frac{\partial G_\alpha}{\partial t}\right|_{\vec{X}_\alpha = \text{constant}} = \frac{DG_\alpha}{Dt}$

→ Here now our objective is to express this Lagrangian change by means of Eulerian approach.

For this we will consider this volume U as a control volume, which is fixed in Eulerian coordinates (i.e. $\vec{x}$). ∴ This control volume is not a one described using material co-ordinates and is not a material volume.

→



Initially at time t, the volume occupied by the fluid continuum is U bounded by surface S. (∴ As it is initial volume, it is also material volume at that stage.)

However in the time interval $\Delta t = t' - t$, this material volume moves and deforms.

It's position and shape at $t' = t + \Delta t$ is shown by dashed lines.

The system at time t consists of volume U and surface S.

The system at time t' will be having U' and S'.

→ In this entire time interval Δt, we can now think of three regions U_I, U_{II}, and U_{III}.

$$\therefore \quad U = U_I + U_{II}$$

$$\text{and} \quad U' = U_{II} + U_{III}$$

So the region U_{II} is common at time t and t'.

⇒ ∴ From the Lagrangian point of view, the temporal rate of change of G_α for this moving system is given by:

$$\left.\frac{DG_\alpha}{Dt}\right|_{system} = \lim_{\Delta t \to 0} \left\{ \frac{\left[(G_\alpha)_{II} + (G_\alpha)_{III}\right]_{t+\Delta t} - \left[(G_\alpha)_I + (G_\alpha)_{II}\right]_t}{\Delta t} \right\}$$

Please recall that

$$(G_\alpha)_I = \int_{U_I} \rho_\alpha \gamma_\alpha \, dU = \int_{U_I} g_\alpha \, dU$$

$$\text{Ily} \quad (G_\alpha)_{II} = \int_{U_{II}} \rho_\alpha \gamma_\alpha \, dU = \int_{U_{II}} g_\alpha \, dU$$

$$\therefore \left.\frac{DG_\alpha}{Dt}\right|_{system} = \lim_{\Delta t \to 0}\left[\frac{(G_\alpha)_{II\,t+\Delta t} - (G_\alpha)_{II\,t}}{\Delta t}\right] + \lim_{\Delta t \to 0}\frac{\left[(G_\alpha)_{III}\right]_{t+\Delta t}}{\Delta t}$$

$$- \lim_{\Delta t \to 0}\frac{\left[(G_\alpha)_{I}\right]_{t}}{\Delta t}$$

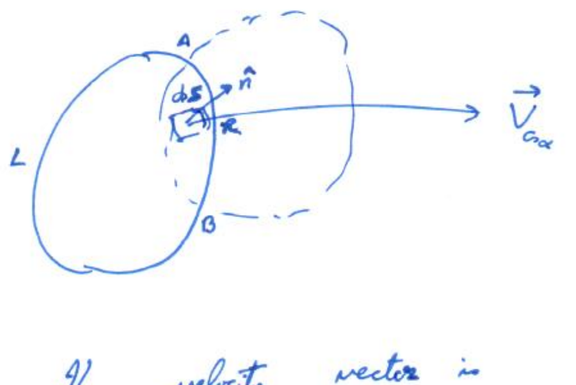
When $\Delta t \to 0$, then volume U_{II} approaches U

$$\therefore \lim_{\Delta t \to 0}\left\{\frac{\left[(G_\alpha)_{II}\right]_{t+\Delta t} - \left[(G_\alpha)_{II}\right]_{t}}{\Delta t}\right\} = \frac{\partial G_\alpha}{\partial t}$$

Now $$\frac{\partial G_\alpha}{\partial t} = \frac{\partial}{\partial t}\int_U \rho_\alpha \gamma_\alpha \, dU = \int_U \frac{\partial}{\partial t}(\rho_\alpha \gamma_\alpha)\, dU$$

Now $$(G_\alpha)_{III} = \int_{U_{III}} \rho_\alpha \gamma_\alpha \, dU$$

i.e. $$\frac{(G_\alpha)_{III}}{\Delta t} \cong \frac{1}{\Delta t}\int_{U_{III}} \rho_\alpha \gamma_\alpha \, dU$$



If velocity vector is $\vec{V}_{G_\alpha}$ and area dS of fluid continuum considered. Then after Δt time, the new change in volume.

$$\frac{1}{\Delta t}\int_{U_{III}} \rho_\alpha \gamma_\alpha \left(\vec{V}_{G_\alpha} \cdot \hat{n}\right) dS\, \Delta t = \frac{(G_\alpha)_{III}}{\Delta t}$$

This is average rate of efflux of the property G_α that has crossed the surface ARB (which is common to U_{II} and U_{III}) during the time interval Δt.

$\Rightarrow$ Similarly $\frac{(G_\alpha)_I}{\Delta t}$ $\rightarrow$ average rate of influx of the property G_α through the surface ALB (which is common to U_I and U_{II})

As $\Delta t \to 0$, $\frac{(G_\alpha)_{III}}{\Delta t}$ and $\frac{(G_\alpha)_I}{\Delta t}$ becomes exact rate of fluxes.

$$\int_{efflux} \rho_\alpha \gamma_\alpha (\vec{V}_{G\alpha} \cdot \hat{n})\, dS$$

$$\int_{influx} \rho_\alpha \gamma_\alpha (\vec{V}_{G\alpha} \cdot \hat{n})\, dS$$

$\therefore$ Net efflux $= \int_S \rho_\alpha \gamma_\alpha (\vec{V}_{G\alpha} \cdot \hat{n})\, dS$

$$\therefore \boxed{\left.\frac{DG_\alpha}{Dt}\right|_{system} = \frac{\partial}{\partial t}\int_U \rho_\alpha \gamma_\alpha\, dU + \int_S \rho_\alpha \gamma_\alpha (\vec{V}_{G\alpha} \cdot \hat{n})\, dS}$$

This is the Reynold's transport theorem used to analyze conservation principles.