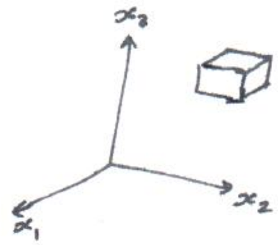
This chapter as discussed in last class is on pressure head and piezometeric head.

Recall for a fluid continuum, the stress on any fluid particle can be expressed as a tensor



$$\sigma_{ij} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix}$$

→ For a liquid (or fluid) we can split it

$$\sigma_{ij} = -p\,\delta_{ij} + \tau_{ij}$$

where p → pressure (a scalar quantity)
δ_{ij} → Kronecker delta
τ_{ij} → viscous stress tensor (including shear stresses)

→ <u>Hydrostatic condition</u>

For hydrostatic conditions, the shear stresses and viscous stresses will be absent

$$\sigma_{ij} = -p\,\delta_{ij}$$

∴ In the force balance equation using

Cauchy's stress principle, we can have

$$\frac{\partial \sigma_{ij}}{\partial x_j} + \rho b_i = 0 \quad ; \quad \begin{array}{l} \rho \rightarrow \text{density of fluid} \\ b \rightarrow \text{Body force per unit mass} \end{array}$$

$$\cancel{-p \frac{\partial \delta_{ij}}{\partial x_j}} \therefore \frac{\partial}{\partial x_j}\left(-p\,\delta_{ij}\right) = -\rho g_i \quad \text{(Hydrostatic conditions)}.$$

Now $g_1 = g_2 = 0$, and $g_3 = -g$

i.e.

$$\frac{\partial p}{\partial x_1} = \frac{\partial p}{\partial x_2} = 0$$

$$\boxed{-\frac{\partial p}{\partial x_3} = \rho g} \qquad \text{Hydrostatic condition.}$$

or

If we replace x_3 by z.

$$-\frac{\partial p}{\partial z} = \rho g$$

or $$z + \frac{p}{\rho g} = \text{constant}$$

The term $\frac{p}{\rho g}$ is pressure head

→ Piezometric Head

In subsurface hydrology while talking about aquifers, the hydrostatic pressure conditions are assumed:

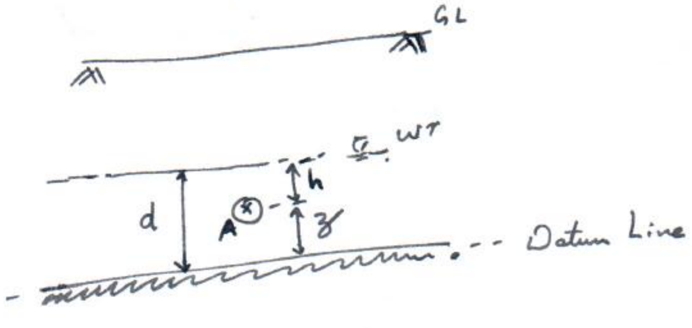


Figure - 2

1) In unconfined aquifers, the water table is having atmospheric pressure p_{atm}.

The pressure of water increases hydrostatically from water table to bottom. Therefore ~~applying~~ evaluating the total energy head at any point in the aquifer (say A in Fig - 2)

$$H_A = \text{Pressure Head} + \text{Velocity Head} + \text{Datum Head}$$

$$= \frac{p_A}{\rho g} + \frac{V_A^2}{2g} + z$$

(Assuming homogeneous liquid or water)

i.e. However, in subsurface hydrology, the velocity of water is very slow and therefore the velocity heads ~~are~~ may be neglected (Not mandatory).

The summation of pressure head and datum head is a more appropriate potential for flow of fluids in porous media.

$$\phi = \frac{p}{\rho g} + z$$

where $\phi \rightarrow$ piezometric head

⇒ In confined aquifers, the piezometric head is again same as $\phi = \frac{p}{\rho g} + z$

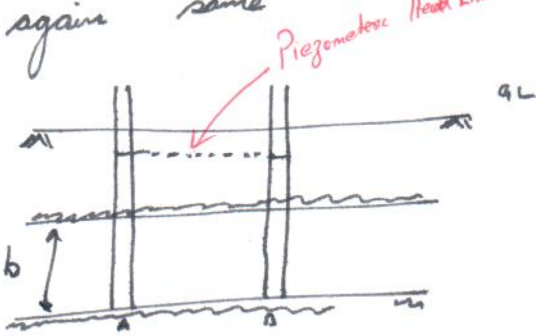


Fig - 3

In two piezometers A and B given, you can connect the water surface that will be above the confining layer. This line is piezometric line.

Chapter – 4

FUNDAMENTAL FLUID TRANSPORT EQUATIONS IN POROUS MEDIA

In the last chapter we discussed the stress tensor for a fluid continuum. How can we utilise the knowledge for porous media flow analysis?

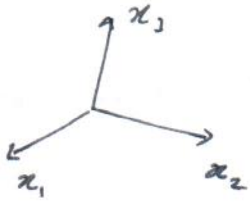
Again the pre-requisite knowledge on continuum fluid mechanics studied in last semester will be useful here.

Velocity : It is the temporal rate of change in position of a fluid particle.
(Fluid actually undergoes continuous deformation when it moves).

$$v_i = \lim_{\delta t \to 0} \frac{\delta s_i}{\delta t}, \quad \text{where } \delta s_i \to \text{displacement vector}$$

Recall we will be always considering now REV and RET for all cases. ∴ $\delta t = 0$ or $\delta U = 0$ cases are not possible. (Note $\delta U \to$ volume of element)

i.e. In a 3D Cartesian coordinate system



$$v_i = f(x_i, t)$$

As the fluid deforms, we need to express the velocity of fluid at various points in the space.

→ A fluid particle is an ensemble of molecules included in the representative elementary volume.

→ Now the properties of REV you may be aware.

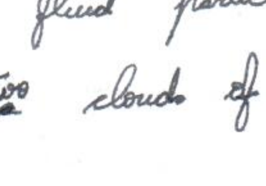
* It may not be a constant value everytime (Especially if the fluid is deforming and the molecules in a particle are moving in and out).

→ So usually when we consider REV or a fluid particle, it should be much larger than mean free path of a single molecule. It should be small to evaluate density at the point. This point is considered as centroid of the REV or particle

Consider a Homogeneous Single Species Fluid

A volume of homogeneous single specie fluid may consist of many REV's or fluid particles.

As told earlier, the molecules in fluids are in continuous random motion. A fluid particle consist of many molecules. Consider two clouds of initially close molecules in two REV's.



As the molecules are in motion, the initial REV are having centroids say x_A and x_B, the molecules may move out. Although if fluid is stationary, the molecules move through the process of molecular diffusion.

→ We have defined REV earlier.

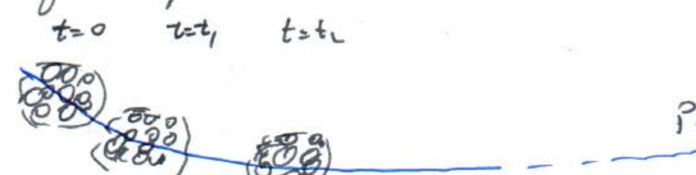
→ If the volume enclosing x_A further expands new molecules from outer side may come in and the existing one goes out.

→ At the same point x_A, we may have to now define a new particle.

As we are considering homogeneous fluid, the new particle at x_A will be having same number of molecules (or mass) as the original particle defined earlier at x_A.

→ Now if the homogeneous fluid is in motion, we may have to re-label a moving particle (as and when the volume of the particle exceeds requisite REV). The end of the path of a former particle now becomes the centroid of newly labeled particle.

★ Thus we can obtain a continuous path of a fluid particle.



That is in actual now we have traced the path of a particle although the molecules encompassing this particle ~~get~~ changes.

Consider Heterogeneous fluid

* The density of the fluid varies.
* So the definitions of fluid particle and velocity of homogeneous fluid may not be applicable.
* Consider there are different species of fluid $\alpha = 1, 2, 3, \ldots$
* If we independently consider each specie α, then the property called particle may be associated for each specie at the same time and same location.

→ Let Consider a volume dU occupied by the multi-species fluid in the 3D cartesian coordinate space.

→ Instantaneous Mass of specie α in the volume $dU = dm_\alpha$

→ Instantaneous Mass of the entire fluid in volume $dU = dm$

Mass density of the α - specie is defined as

$$\rho_\alpha = \frac{\text{Mass of } \alpha\text{-specie}}{\text{Volume of multi-species fluid}} = \frac{dm_\alpha}{dU}$$

If there are total N - species

i.e. $\alpha = 1, 2, 3, \ldots, N$

then $$\sum_{\alpha=1}^{N} \rho_\alpha = \sum_{\alpha=1}^{N} \left(\frac{dm_\alpha}{dU} \right) = \frac{1}{dU} \sum_{\alpha=1}^{N} (dm_\alpha)$$

i.e. $\frac{1}{dU} \sum_{\alpha=1}^{\tilde{N}} (dm_\alpha) = \frac{dm}{dU} = \rho$ (density of fluid).

* The velocity distribution and pathlines are different for each species and also different from the entire fluid system

* Define <u>Mass fraction</u> $\omega_\alpha = \frac{\rho_\alpha}{\rho}$

$$\therefore \sum_{\alpha=1}^{\tilde{N}} \omega_\alpha = 1$$

* Velocity $\vec{V}_\alpha$ (we are using fixed co-ordinate system)
  → It is the statistical average velocity of the individual molecules of the α - species. within the volume dU.

$$\left(= \frac{\text{Sum of velocities of specie } \alpha}{\text{Number of molecules of specie } \alpha} \right)$$

→ Please note that now $\vec{V}_\alpha$ is defined at fluid continuum level approach. (Microscopic level).

* We can also define Mass Average Velocity $\vec{V}^*$

$$\vec{V}^* = \frac{\left(\sum_{\alpha=1}^{\tilde{N}} \rho_\alpha \vec{V}_\alpha \right)}{\sum_{\alpha=1}^{\tilde{N}} \rho_\alpha} = \frac{\sum_{\alpha=1}^{\tilde{N}} (\rho_\alpha \vec{V}_\alpha)}{\rho}$$

Q: What is this quantity $\rho_\alpha \vec{V}_\alpha$?

→ This quantity $\rho_\alpha \vec{V}_\alpha$ describes the mass flux of specie α.

(Mass of specie α passing through a unit area per unit time. This area is placed perpendicularly now to the flow).

$\therefore \sum_{\alpha=1}^{\tilde{n}} \rho_\alpha \vec{V}_\alpha$ → gives the mass flux of the entire fluid system

$= \rho \vec{V}^*$

Note: When you are using a pitot tube in a non-homogeneous liquid, it measures actually $\vec{V}^*$.

The <u>interpretation of $\vec{V}^*$</u> : → It is the momentum per unit ~~mass~~ mass of the flowing fluid.

$\rho \vec{V}^*$ → Momentum per unit volume

$\therefore \sum_{\alpha=1}^{\tilde{n}} \rho_\alpha \vec{V}_\alpha = \rho \vec{V}^*$

Sum of momenta of individual species (per unit volume). = Momentum per unit volume