UNSATURATED FLOW ....

In the last class, we started discussing on unsaturated flows. A soil is said to be unsaturated if all the pores in the media are not filled with water.
Using the conventional Reynolds Transport Theorem (control volume approach) we arrived at the one-dimensional (only vertical direction) continuity equation

$$\frac{\partial \theta}{\partial t} + \frac{\partial q_z}{\partial z} = 0 \quad \rightarrow (1)$$

where $\theta \rightarrow$ moisture content that may change w.r.t. time and space.

$q_z \rightarrow$ Unsaturated specific discharge in z-direction.

Recall in saturated porous media, the specific discharge was given as:

$$q_z = -K \frac{\partial \phi}{\partial z}$$

where K is hydraulic conductivity in saturated media (Ofcourse assuming the porous media is isotropic).

⇒ The same principle can be adopted for unsaturated porous media, however, there is some difference

$$q_z = -K(\theta) \frac{\partial \phi}{\partial z} \quad \rightarrow (2)$$

where $K(\theta)$ is unsaturated hydraulic conductivity,
$\phi \rightarrow$ hydraulic head.

Substituting (2) in equation (1), we will get

$$\frac{\partial \theta}{\partial t} + \frac{\partial}{\partial z} q_z = 0$$

i.e. $$\frac{\partial \theta}{\partial t} + \frac{\partial}{\partial z}\left(-K(\theta)\frac{\partial \phi}{\partial z}\right) = 0$$

Recall in last class, we discussed that the hydraulic head $\phi = \psi + z$

(In unsaturated conditions, water particles may strongly attach to the soil grains due to surface tension). $\psi \rightarrow$ suction head, $z \rightarrow$ Datum head

$$\therefore \frac{\partial \theta}{\partial t} + \left[-\frac{\partial K}{\partial z}\frac{\partial \phi}{\partial z} - K\frac{\partial^2 \phi}{\partial z^2}\right] = 0 \rightarrow (3)$$

Considering $$q_z = -K(\theta)\frac{\partial}{\partial z}(\psi + z)$$

$$= -K(\theta)\frac{\partial \psi}{\partial z} - K(\theta)$$

In equation (3), we have dependent variables θ and ϕ and K. If we can define any constitutional relations between θ, ϕ and K, we can convert equation (3) in terms of one unknown.

Let us consider that suction ~~head~~ is dependent on moisture content. That is, when the soil is fully saturated there will be no suction and when the soil is dry, there will be maximum suction.

Various scientists (Brooks and Corey, 1964), Gardner (1958), van Genuchten (1980) etc. have tried to develop constitutional relations between θ and ψ.

Using these relations, there exist the gradients $\frac{d\theta}{d\psi}$. This gradient $\frac{d\theta}{d\psi}$ is called specific water capacity.

$\therefore$ We can now describe $\frac{\partial \psi}{\partial z} = \frac{d\psi}{d\theta} \cdot \frac{\partial \theta}{\partial z}$

i.e. $$\frac{\partial \psi}{\partial z} = \frac{d\psi}{d\theta} \cdot \frac{\partial \theta}{\partial z}$$

$$\therefore \quad q_z = -K(\theta) \frac{d\psi}{d\theta} \cdot \frac{\partial \theta}{\partial z} - K(\theta) \rightarrow (4)$$

Equation (1) is now:

$$\cancel{\frac{\partial \theta}{\partial t} + \left[-\frac{\partial K}{\partial z} \right.} \quad \frac{\partial \theta}{\partial t} + \frac{\partial}{\partial z}\left[-K(\theta) \frac{d\psi}{d\theta} \cdot \frac{\partial \theta}{\partial z} - K(\theta) \right] = 0 \rightarrow (5)$$

We can also define the quantity $K(\theta) \frac{d\psi}{d\theta}$ as soil water diffusivity $D(\theta)$

i.e. $D = K(\theta) \frac{d\psi}{d\theta}$

Equation 5 becomes

$$\frac{\partial \theta}{\partial t} + \frac{\partial}{\partial z}\left[- D(\theta)\frac{\partial \theta}{\partial z} - K(\theta) \right] = 0$$

or
$$\boxed{\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial z}\left(D(\theta) \frac{\partial \theta}{\partial z} + K(\theta) \right)} \rightarrow (6)$$

Equation 6 is one-dimensional Richard's equation to describe unsaturated flow.

⇒ Solving equation 6, we can get the distribution of θ.

However in the present form, still $K(\theta)$ is a dependent variable.

We need to describe again constitutional relation between K and θ.

This relation should be such that when the soil is fully saturated the hydraulic conductivity will be a constant value ~~of~~ say K_s (saturated hydraulic conductivity).

→ Similarly, hydraulic conductivity will be zero when the moisture content is near about zero.

We can express $K(\theta) = K_r(\theta)\, K_s$

where $K_r(\theta)$ is a relative term $0 \le K_r(\theta) \le 1.0$.

Equation 6 is one-dimensional Richard's equation. We can extend the concept in three dimensions such that:

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial x_i}\left(D_{ij}(\theta) \frac{\partial \theta}{\partial x_j} + K_{ij}(\theta) \right)$$

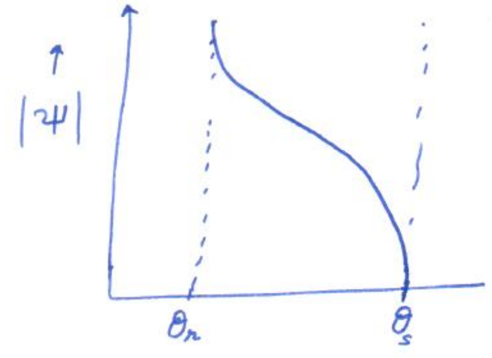
where $D_{ij}(\theta) \rightarrow$ soil water diffusivity tensor. and in function of moisture content θ.

$K_{ij}(\theta) \rightarrow$ unsaturated hydraulic conductivity tensor

$$D_{ij}(\theta) = K_{ij}(\theta) \frac{d\psi}{d\theta}$$

$$= K_r(\theta)\, K_{s_{ij}} \frac{d\psi}{d\theta}$$

→ The constitutional relations between θ and ψ can be figuratively represented as:



This curve is described by various scientists using empirical equations (SWCC)

→ Soil Water Characteristic Curves

(Brooks-Corey, Gardner, van Genuchten, etc.)

→ These soil scientists also have described effective water content

$$\theta_e = \frac{\theta - \theta_r}{\theta_s - \theta_r}$$

where $\theta_e \rightarrow$ effective water content

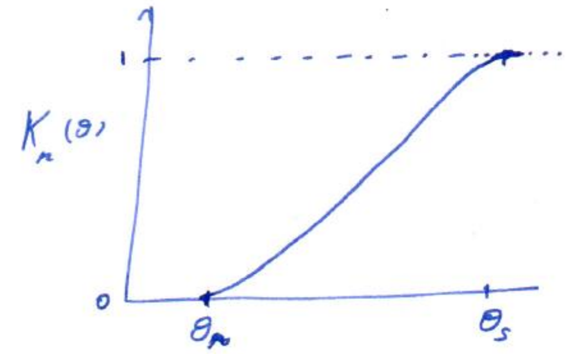
$\theta_s \rightarrow$ saturated water content

$\theta_r \rightarrow$ residual moisture content

(This is the minimum water content in the soil, that ~~may~~ is not possible to be drained).

$\Rightarrow$ Similarly we can also correlate $K(\theta)$ with respect to θ.

Recall $K(\theta) = K_r(\theta)\, K_s$



Again the earlier mentioned scientists have given empirical equations ~~to~~ to describe $K_r(\theta)$ with respect to θ.

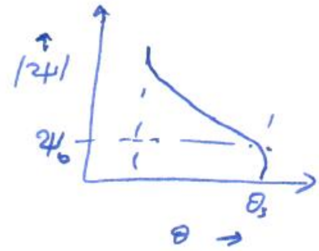
$\Rightarrow$ The Brooks - Corey constitutional relationship is

(i) $$\theta_e = \frac{\theta - \theta_r}{\theta_s - \theta_r} = \left(\frac{\psi_b}{\psi}\right)^{\lambda}$$

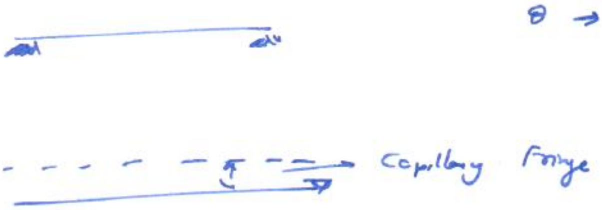
where $\theta_e \rightarrow$ already described

$\psi_b \rightarrow$ bubbling capillary pressure.

$\lambda \rightarrow$ pore-size index for B.C.'s relations



From the graph, you can see that due to capillary rise there will be capillary fringe where pores are filled with water. Beyond capillary fringe, air enters the pores and this is bubbling pressure.



Similarly B.C.'s K vs θ relations:

$$\frac{K(\theta)}{K_s} = K_r(\theta) = \left(\theta_e\right)^n$$

$$n = 3 + \frac{2}{\lambda}$$

2) Van-Genuchten: relation

$$\theta_e = \left[\frac{1}{1 + (\alpha|\psi|)^n}\right]^m \quad ; \quad \begin{aligned} \alpha &= \text{parameter constant} \\ n &= \text{constant} \\ m &= \text{constant} \end{aligned}$$