LECTURE 30

28-MAR-2014

## Continuity Equation for Unconfined Flow using Dupuit's Approximations

Yesterday, we discussed the Dupuit's approximations for unconfined flow.

→ In an $x$-$z$ plane, the flow is present only in $x$-direction

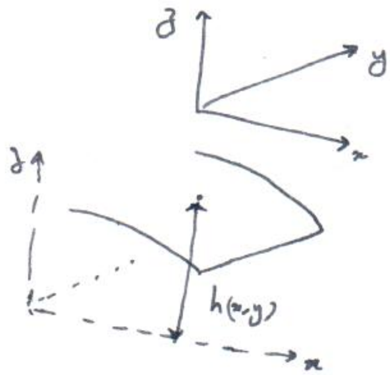
→ Now the same concept can be extended to $x$-$y$-$z$ space, in which the flow is present only in $x$ and $y$ directions. ( i.e. Only horizontal flow).

So from Dupuit's approximations we have

$$q_x = -K \frac{dh}{dx}$$

$$q_y = -K \frac{dh}{dy}$$

where $h = h(x,y)$




In transient conditions, we have for the isotropic porous media, the unconfined flow variables

$q_x(x,y,t)$, $q_y(x,y,t)$, and $h(x,y,t)$.

$$q_x = -K \frac{dh}{dx} \quad \text{and} \quad q_y = -K \frac{dh}{dy}$$

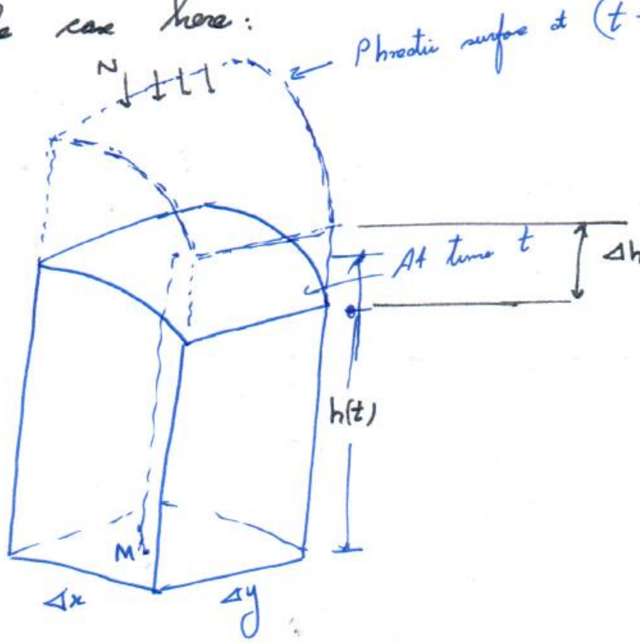
are two momentum equations. We require one more equation to solve for three unknowns. ∴ Continuity equation is employed.

Recall in one of our earlier chapters we had described the continuity equation for a homogeneous fluid as:

$$\nabla \cdot (\rho \vec{q}\,) + \frac{\partial}{\partial t}(\rho n) = 0$$

where $\rho \vec{q} \rightarrow$ Mass flux (Mass per unit area per unit time)

In the case here:




Consider the phreatic surfaced aquifer control volume.

This control volume is prepared for a mathematical point M in a horizontal plane $(x, y)$ i.e. $M(x,y)$.

The elevation of phreatic surface at this point M is $h$.

We have at time '$t$', $h = h(t)$.

Again utilising RTT

$$\frac{DB}{Dt} = \frac{\partial}{\partial t}\iiint \beta \rho \, dU + \iint_{cs} \beta \rho (\vec{V}\cdot\hat{n})\, dA$$

Here $B$ = mass of water, $\beta = 1$, $\frac{DB}{Dt} = 0$

$\iint_{cs} \rho(\vec{V}\cdot\hat{n})\, dA$ = Net mass outflow through control surfaces of the volume.

Unlike the confined aquifer, here the phreatic surface changes w.r.t. time. That is the control volume changes w.r.t time.

⇒ To employ Dupuit's approximations and to develop continuity equation: it is better to consider about mass fluxes and generations over an ~~p~~ interval of time (say $\Delta t$) rather than instantaneously.

⇒ At the instant 't', for the mathematical location $(x, y)$ the water elevation is $h(t)$.

We created a control volume within the span

$$x - \frac{\Delta x}{2} \text{ to } x + \frac{\Delta x}{2} \quad \text{and} \quad y - \frac{\Delta y}{2} \text{ to } y + \frac{\Delta y}{2}.$$

⇒ The mass flux that comes into control volume in $x$- direction at time 't' is

$$= \rho q_x\left(x - \frac{\Delta x}{2}, y\right)$$

But it is difficult to interpret Net mass outflux in x- direction at time 't'.

∴ The mass of water that enters into the CV in $x$- direction at ~~time~~ the instant 't'

$$= \rho q_x\left(x - \frac{\Delta x}{2}, y\right) h(t) \Delta y$$

(Note: Due to Dupuit's assumptions, the aquifer depth is quite large and the difference of $h$ at $(x - \frac{\Delta x}{2}, y)$ ~~at~~ and at $(x + \frac{\Delta x}{2}, y)$ are negligible).

Let us define $q_x h = Q_{w_x}$ discharge per unit width in x- direction

$\therefore$ Mass of water coming in ~~a~~ the instant time 't'

$$= \rho\, Q_{w(x-\frac{\Delta x}{2},y)} \Delta y + \rho\, Q_{w(y-\frac{\Delta y}{2},x)} \Delta x$$

Again we can now approximate (as $\Delta x$ and $\Delta y$ are small)

Mass of water that comes in time interval $\Delta t$

$$= \rho\, Q_{w(x-\frac{\Delta x}{2},y)} \Delta y\, \Delta t + \rho\, Q_{w(x,y-\frac{\Delta y}{2})} \Delta x\, \Delta t$$

$\therefore$ Net mass of water that goes out from CV in time interval $\Delta t$,

$$= \left[\rho\, Q_{w(x+\frac{\Delta x}{2},y)} - \rho\, Q_{w(x-\frac{\Delta x}{2},y)}\right] \Delta y\, \Delta t + \left[\rho\, Q_{w(x,y+\frac{\Delta y}{2})} - \rho\, Q_{w(x,y-\frac{\Delta y}{2})}\right] \Delta y\, \Delta t$$

In z- direction there is accretion of water into the aquifer at a rate $-N$ $[L T^{-1}]$ in the time interval $\Delta t$

$\therefore$ Mass Inflow from accretion in the time interval $\Delta t$

$$= \rho N \Delta x \Delta y \Delta t$$

As per RTT: now

$$\frac{\partial}{\partial t} \iiint \rho\, dU_w + \iint_{cs} \rho(\vec{V}\cdot\hat{n})\, dA = 0$$

or

$$\Delta\left[\iiint \rho\, dU_w\right] + \left[\iint \rho(\vec{V}\cdot\hat{n})\, dA\right]\Delta t = 0$$

~~$\iint_{cs} \rho(\vec{V}\cdot\hat{n})\, dA$~~

~~$\therefore$ The net mass outflow in time interval $\Delta t$~~

Now we can say that:

$$\left[\text{Change in storage in CV}\right] + \left[\text{Net Mass Outflow in time interval } \Delta t\right] = 0$$

Net Mass Outflow in time interval $\Delta t$

$$= \left[\rho\, Q_{w_{(x+\frac{\Delta x}{2},\,y)}} - \rho\, Q_{w_{(x-\frac{\Delta x}{2},\,y)}}\right] \Delta y\, \Delta t + \left[\rho\, Q_{w_{(x,\,y+\frac{\Delta y}{2})}} - \rho\, Q_{w_{(x,\,y-\frac{\Delta y}{2})}}\right] \Delta x\, \Delta t$$

$$- \rho N\, \Delta x\, \Delta y\, \Delta t$$

For Change in storage in CV:

(Recall concept of Specific Yield).

$$\text{Change in storage} = \rho\, n_e\, \Delta x\, \Delta y\left[h(t+\Delta t) - h(t)\right] + \int_0^h \left[(\rho n)_{t+\Delta t} - (\rho n)_t\right] \Delta x\, \Delta y\, dz$$

(Please note the porous media may be compressible and therefore the second term).

We have

$$\rho\, n_e\, \Delta x\, \Delta y\left[h_{t+\Delta t} - h_t\right] + \int_0^h \left[\rho n_{t+\Delta t} - \rho n_t\right] dz\, \Delta x\, \Delta y$$

$$+ \left[\rho\, Q_{w_{(x+\frac{\Delta x}{2},\,y)}} - \rho\, Q_{w_{(x-\frac{\Delta x}{2},\,y)}}\right] \Delta y\, \Delta t + \left[\rho\, Q_{w_{(x,\,y+\frac{\Delta y}{2})}} - \rho\, Q_{w_{x,\,y-\frac{\Delta y}{2}}}\right] \Delta x\, \Delta t$$

$$- \rho N\, \Delta x\, \Delta y\, \Delta t = 0$$

Dividing throughout by $\Delta x\, \Delta y\, \Delta t$:

$$\frac{\rho\, n_e\left[h_{t+\Delta t} - h_t\right]}{\Delta t} + \frac{1}{\Delta t}\int_0^h (\rho n_{t+\Delta t} - \rho n_t)\, dz + \frac{\left[\rho\, Q_{w_{(x+\frac{\Delta x}{2},\,y)}} - \rho\, Q_{w_{(x-\frac{\Delta x}{2},\,y)}}\right]}{\Delta x}$$

$$+ \frac{\left[\rho\, Q_{w_{x,\,y+\frac{\Delta y}{2}}} - \rho\, Q_{w_{x,\,y-\frac{\Delta y}{2}}}\right]}{\Delta y} - \rho N = 0$$

Taking limits: we will get: (i.e. $\Delta x \to 0$, $\Delta y \to 0$, $\Delta t \to 0$)

$$\boxed{\rho n_e \frac{\partial h}{\partial t} + \int_0^h \frac{\partial(\rho n)}{\partial t}\, dz + \frac{\partial}{\partial x}(\rho Q_{w_x}) + \frac{\partial}{\partial y}(\rho Q_{w_y}) - \rho N = 0}$$

However, $\int_0^h \frac{\partial}{\partial t}(\rho n)\, dz = \int_0^h S^*_{op} \frac{\partial p}{\partial t}\, dz$

where $S^*_{op} \to$ specific mass storativity related to pressure changes

Recall $S^*_{op} = \rho\left[\beta n + \alpha(1-n)\right]$

where $\beta \to$ water compressibility
$\alpha \to$ soil matrix compressibility
$n \to$ porosity

Also from Dupuit's assumptions: $\frac{\partial p}{\partial t} = \rho g \frac{\partial h}{\partial t}$

$$\therefore \int_0^h \frac{\partial}{\partial t}(\rho n)\, dz = \rho g\, S^*_{op}\, h \frac{\partial h}{\partial t}$$

$\therefore$ Continuity equation becomes:

$$-\frac{\partial}{\partial x}(\rho Q_{w_x}) - \frac{\partial}{\partial y}(\rho Q_{w_y}) + \rho N = \rho\left(n_e + S^*_{op}\, g h\right)\frac{\partial h}{\partial t}$$

In unconfined aquifers, in most cases:

$$\rho n_e >\!>\!> \rho S^*_{op}\, g h$$

$\therefore$ We can write: $\cancel{\frac{\partial}{\partial x}(\rho Q_{w_x}) - \frac{\partial}{\partial y}\rho Q_{w_y}}$

$$-\frac{\partial}{\partial x}(\rho\, q_{wx}) - \frac{\partial}{\partial y}(\rho\, q_{wy}) + \rho N = \rho\, n_e \frac{\partial h}{\partial t}$$

For incompressible fluids; this becomes:

$$-\frac{\partial}{\partial x}(q_{wx}) - \frac{\partial}{\partial y}(q_{wy}) + N = n_e \frac{\partial h}{\partial t}$$

Substituting $q_{wx} = -K h \frac{\partial h}{\partial x}$, etc. for isotropic aquifers:

$$\boxed{\frac{\partial}{\partial x}\left(K h \frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial y}\left(K h \frac{\partial h}{\partial y}\right) + N = n_e \frac{\partial h}{\partial t}}$$

This is the governing equation for unconfined flow in aquifers.

Also this eqn. becomes

$$\boxed{\frac{K}{2}\left[\frac{\partial^2 (h^2)}{\partial x^2} + \frac{\partial^2 (h^2)}{\partial y^2}\right] + N = n_e \frac{\partial h}{\partial t}}$$

This is Forchheimer's equation.