LECTURE 29

27-MAR-2014

UNCONFINED FLOW

→ The meaning of unconfined flows is:
  * Presence of phreatic surface that bounds flow

→ In nature, we have unconfined aquifers.

→ The phreatic surface may be changing w.r.t time at a location

  ∴ If we want to fix boundary for saturated porous medium, then the upper boundary (i.e. phreatic surface) may not be fixed.

→ To solve unconfined flow problems, we require lots of assumptions

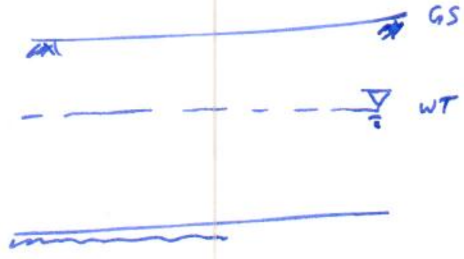
A famous approximation is Dupuit's assumptions.

Dupuit Assumptions

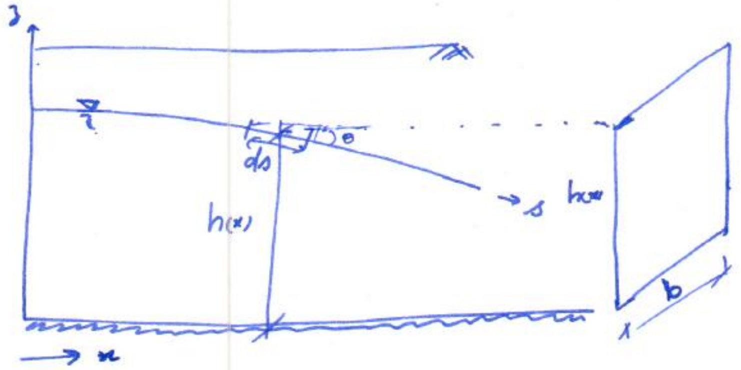
Based on number of observations Dupuit (1863) formulated some assumptions that are very much applicable to slow groundwater flows:

* Water table or phreatic surface is only slightly inclined.
* Streamlines for unconfined flow may be assumed horizontal
* Slopes of the phreatic surface and hydraulic gradients are almost equal.

For a 2D unconfined flow without accretion in steady state condition



* We consider a vertical plane xz for analysis.
* The phreatic surface is a streamline.



Assuming isotropic aquifer:

As the phreatic surface is a streamline, the specific discharge along streamline can be evaluated.

$$q_s = -K \frac{d\phi}{ds}$$

As $\phi = \frac{p}{\rho g} + z$

and $p = 0$ along phreatic surface,

we have $\frac{d\phi}{ds} = \frac{dz}{ds}$

$$\therefore q_s = -K \frac{dz}{ds}$$

From the figure, it is clear $\frac{dz}{ds} = \sin\theta$

$\therefore \quad q_x = -K \sin\theta$

However from the assumption that water table or phreatic surface is only slightly inclined, we can write

$$\sin\theta \approx \tan\theta = \frac{dh}{dx} \quad \text{(the slope of phreatic surface).}$$

Now we have $q_x = -K \frac{dh}{dx}$

(Suggesting the flow is essentially horizontal).

⇒ If there is width 'b' as shown in figure, then the total discharge is given as:

$$Q_x = -K\,b\,h(x)\,\frac{dh}{dx} \quad \rightarrow (1)$$

⇒ Therefore using Dupuit's assumption has helped in reducing a two-dimensional problem to a one-dimensional in x-direction.

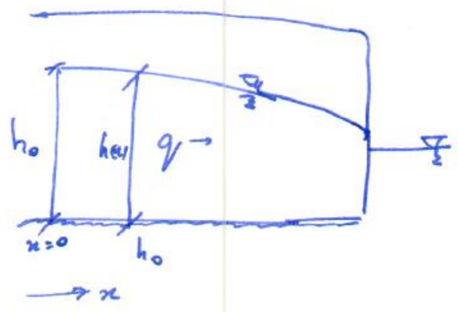
Equation (1) is a steady unconfined groundwater flow equation:

→ On integrating (1) we will be able to solve for h.

Case 1: Hydraulic steady flows in Homogeneous Media:

Let Q_w be discharge per unit width $[L^3T^{-1}L^{-1}]$

$\therefore \quad Q_w = q \, h(x)$



$$Q = -K h \frac{dh}{dx} = \text{constant}$$

(steady ~~d~~ discharge is assumed for this case)

$$\therefore \quad Q\,dx = -K h \, dh$$

$$\text{or} \quad Q \int_0^x dx = -K \int_{h_0}^{h} h'\,dh'$$

$$\cancel{Q = -K \; h_0^2 - h_L^2}$$

$$Q = \frac{-K\left(h_0^2 - h^2\right)}{2x}$$

$$\text{or} \quad \boxed{h^2 = h_0^2 - \frac{2Qx}{K}}$$

If $h = h_L$ at $x = L$, and $h = h_0$ at $x = 0$

Then

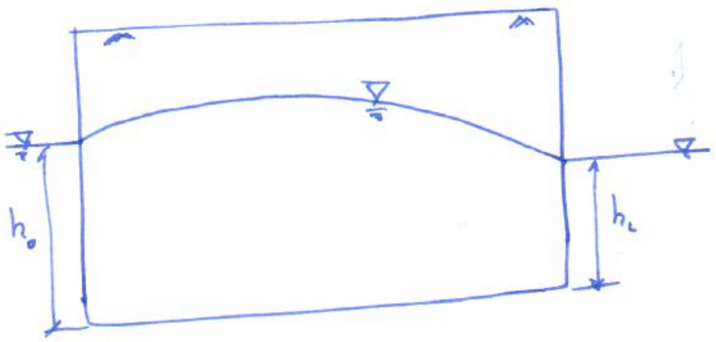
$$Q = \frac{K\left(h_0^2 - h_L^2\right)}{2L}$$

$$\text{or} \quad h_L^2 = h_0^2 - \frac{2QL}{K}$$

This is Dupuit - Forchheimer discharge formula.

Problem

Two rivers are located 1000 m apart. In between these two rivers lie a phreatic aquifer as shown in figure



The aquifer hydraulic conductivity is $K = 0.5$ m/d. The region recieves an average rainfall of 15 cm/yr and evaporation of 10 cm/yr. Water elevation in river on left side is 20 m. Water elevation in river on right side is 18 m. Determine the discharge per unit width in the unconfined aquifer.

Solution.

Note: we had suggested discharge per unit width

$$= \frac{Q}{b} = Q_w$$

$$Q_w = \frac{K}{2L}\left(h_0^2 - h_L^2\right)$$

(Please Note that we are considering the situation to be in steady conditions. That is the discharge in the aquifer is constant based on constant water elevations h_0 and h_L at the two boundaries).

$L = 1000$ m, $h_0 = 20$ m, $h_L = 18$ m

$$Q_w = \frac{0.5}{2\times 1000}\left[20^2 - 18^2\right] = \frac{38}{2000} \ m^2/d$$

Please note we have not incorporated the net recharge rate.

If net recharge rate = $15 - 10 = 5$ cm/yr.

⇒ From continuity we have for recharge situation

$$Q_w = -Kh\frac{dh}{dx} = Wx$$

where W = recharge rate $[L T^{-1}]$.

Solving, $$h^2 = h_L^2 + \frac{W}{K}\left(L^2 - x^2\right)$$

where h → head at any distance x,

h_L → water elevation at right boundary

h_0 → water elevation at left boundary
