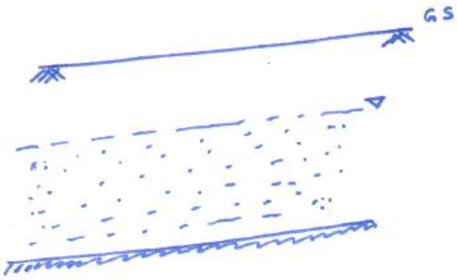
LECTURE 28
24-MAR-2014

Phreatic Surface without Accretion (Steady State)

In the last class we were discussing on initial and boundary value conditions for groundwater hydrological problems:

→ After discussing ~~with~~ about the three B.C.'s
* Dirichlet B.C. ; Neumann B.C. ; Mixed B.C.
* The cases of prescribed potential, no-flow boundary conditions, etc. were discussed
* Subsequently we started describing the boundary condition for phreatic surface (or water table) in steady state condition.



Recall, the water pressure on water table, $p = 0$

$\therefore$ In steady conditions

$$\phi = \phi(x, y, z)$$

$$\phi = \frac{p}{\rho g} + z \quad \text{or} \quad \boxed{\phi(x, y, z) - z = 0}$$

$\therefore$ We can assign $F(x, y, z) = 0 = \phi(x, y, z) - z$

Again $\nabla F = \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \left(\frac{\partial \phi}{\partial z} - 1\right)\hat{k} = 0$

⇒ The peculiarity of water table or gas-liquid interface is that there will not be any flow across this interface. Therefore this interface is similar to no-flow boundary.

i.e. $\vec{q} \cdot \hat{n} = 0$

Recall we had obtained $\hat{n} \rightarrow$ the unit outward normal

$$\hat{n} = \frac{\nabla F}{|\nabla F|} = \frac{1}{|\nabla F|}\left[\frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \left(\frac{\partial \phi}{\partial z} - 1\right)\hat{k}\right]$$

Also:

$$\vec{q} = -\begin{pmatrix} K_x & 0 & 0 \\ 0 & K_y & 0 \\ 0 & 0 & K_z \end{pmatrix}\begin{Bmatrix} \frac{\partial \phi}{\partial x}\hat{i} \\ \partial\phi/\partial y\,\hat{j} \\ \partial\phi/z\,\hat{k} \end{Bmatrix}$$

$$\therefore \vec{q} \cdot \hat{n} = 0 = \left(-K_x \frac{\partial \phi}{\partial x}\frac{\partial \phi}{\partial x}\right) + \left(-K_y \frac{\partial \phi}{\partial y}\frac{\partial \phi}{\partial y}\right) + \left(-K_z \frac{\partial \phi}{\partial z}\frac{\partial \phi}{\partial z}\right) - \left(-K_z \frac{\partial \phi}{\partial z}\right)$$

i.e. The boundary condition for <u>steady phreatic surface without accretion</u> can be given as:

$$\boxed{K_x\left(\frac{\partial \phi}{\partial x}\right)^2 + K_y\left(\frac{\partial \phi}{\partial y}\right)^2 + K_z\left(\frac{\partial \phi}{\partial z}\right)^2 - K_z\frac{\partial \phi}{\partial z} = 0}$$

If the unconfined aquifer in this case is isotropic, then $K_x = K_y = K_z = K$

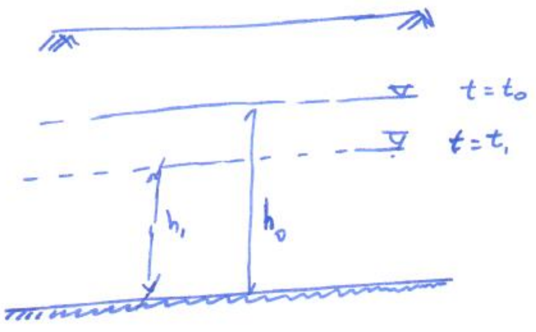
The B.C. of phreatic surface will be then:

$$\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2 + \left(\frac{\partial \phi}{\partial z}\right)^2 - \frac{\partial \phi}{\partial z} = 0$$

$\Rightarrow$ Also note that as there is no flow across water table, the water table becomes a streamline

i.e. Streamfunction ψ = constant along water table.

Unsteady phreatic surface without accretion



→ Let us consider a situation, where the phreatic surface of unconfined aquifer is in unsteady condition.

→ An assumption involved is that the capillary fringe is neglected.

→ The unsteady free surface without accretion is a material or fluid surface always consisting of same fluid particles.

* The free surface is represented by

$$F(x, y, z, t) = 0$$

* As informed this fluid free surface always consists of same fluid particles (because of non-accretion).

$\therefore$ Material derivative : $\dfrac{DF}{Dt} = 0$

i.e. $\dfrac{\partial F}{\partial t} + (\vec{V} \cdot \nabla) F = 0$

The velocity of a fluid particle that belongs to the free surface, $\vec{q} = n\vec{V}$

or $\vec{q} = n_e \vec{V}$; if $n_e \neq n$

where $n_e \rightarrow$ effective porosity.

$$\therefore \quad \frac{\partial F}{\partial t} + \frac{\vec{q}}{n_e} \cdot \nabla F = 0$$

The free surface equation is $\phi(x,y,z,t) - z = 0$

Here again, as described in earlier section, we need to utilise the terms ∇F, $\hat{n}$ and take $\frac{\partial F}{\partial t}$

We have the relation $\vec{q} = -\underline{\underline{K}} \cdot \nabla \phi$

$$\frac{\partial F}{\partial t} + \frac{1}{n_e}\left(-\underline{\underline{K}} \cdot \nabla \phi\right) \cdot \nabla F = 0 \qquad \longrightarrow (1)$$

$$\nabla F = \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \left(\frac{\partial \phi}{\partial z} - 1\right)\hat{k}$$

$\therefore$ Equation (1) becomes

$$0 = \frac{\partial \phi}{\partial t} + \frac{1}{n_e}\left[-K_x\left(\frac{\partial \phi}{\partial x}\right)^2 - K_y\left(\frac{\partial \phi}{\partial y}\right)^2 - K_z\left(\frac{\partial \phi}{\partial z}\right)^2 + K_z \frac{\partial \phi}{\partial z}\right]$$

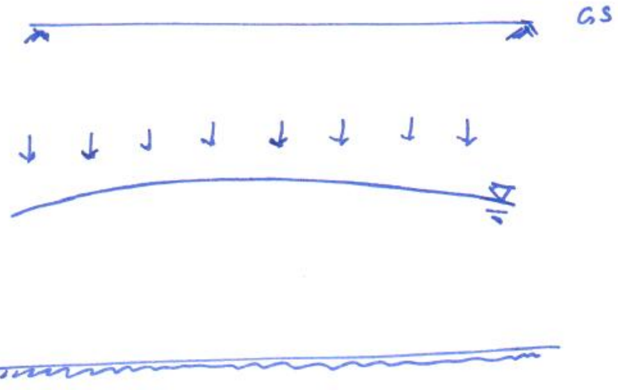
For isotropic porous medium:

$$0 = \frac{\partial \phi}{\partial t} - \frac{K}{n_e}\left[\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2 + \left(\frac{\partial \phi}{\partial z}\right)^2 - \frac{\partial \phi}{\partial z}\right]$$

Steady phreatic surface with accretion

Q: What is meant by accretion?

It is the rate at which liquid is added to an aquifer or saturated region at its free surface.



Recall for steady phreatic surface with no accretion, we had

$$\vec{q} \cdot \hat{n} = 0$$

⇒ Let us consider here that water accretion is occuring only in vertical direction. (i.e. in the z-direction).

* As the vertical direction is +ive upwards, then if water is infiltrating into the water table, we have accretion as negative value say $(-N)$.
* If there is evaporation (water removed from water table), then we have $+N$ in units of $[L\,T^{-1}]$.

For infiltration of $-N$ in the vertical direction we can represent this infiltration as a vector (Opposite for evaporation).

$$\vec{N} = -N\,\hat{k}$$

We have water table surface as shown below:

∴ Flux across water table

$$\vec{q} \cdot \hat{n} = \vec{N} \cdot \hat{n}$$

$\therefore$ We have $(\vec{q} - \vec{N}) \cdot \hat{n} = 0$

Recall $\vec{q} = \left(-K_x \frac{\partial \phi}{\partial x}\right)\hat{i} + \left(-K_y \frac{\partial \phi}{\partial y}\right)\hat{j} + \left(-K_z \frac{\partial \phi}{\partial z}\right)\hat{k}$

$$\hat{n} = \frac{1}{|\nabla F|}\left[\frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \left(\frac{\partial \phi}{\partial z} - 1\right)\hat{k}\right]$$

Now we have $(\vec{q} - \vec{N}) \cdot \hat{n} = 0$ so:

$$\frac{1}{|\nabla F|}\left[-K_x\left(\frac{\partial \phi}{\partial x}\right)^2 - K_y\left(\frac{\partial \phi}{\partial y}\right)^2 - K_z\left(\frac{\partial \phi}{\partial z}\right)^2 + K_z \frac{\partial \phi}{\partial z}\right] + \frac{N}{|\nabla F|}\left(\frac{\partial \phi}{\partial z} - 1\right) = 0$$

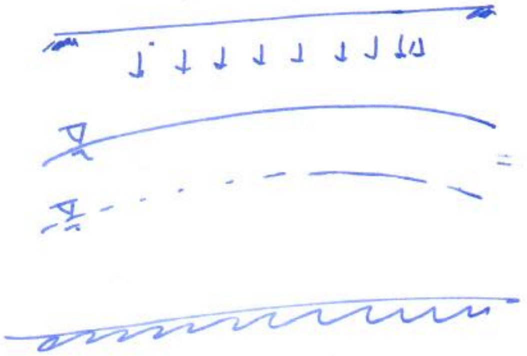
i.e. $$\boxed{K_x\left(\frac{\partial \phi}{\partial x}\right)^2 + K_y\left(\frac{\partial \phi}{\partial y}\right)^2 + K_z\left(\frac{\partial \phi}{\partial z}\right)^2 - K_z \frac{\partial \phi}{\partial z} - N\left(\frac{\partial \phi}{\partial z} - 1\right) = 0}$$

$\Rightarrow$ If you have infiltration or accretion from all three directions, then accretion vector

$$\vec{N} = N_x \hat{i} + N_y \hat{j} + N_z \hat{k}$$

You have to appropriately form the boundary condition equation:

Unsteady Free Surface With Accretion



Let there be accretion in the vertical direction

$\vec{N} = -N\hat{k}$ (for infiltration).

The piezometric head

$$\phi = \frac{p}{\rho g} + z$$

As described earlier, on the free surface:

$$\phi(x, y, z, t) - z = 0$$

Also $\rho g \frac{\partial \phi}{\partial t} = \frac{\partial p}{\partial t} + \rho g \frac{\partial z}{\partial t}$

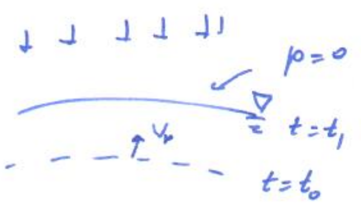
$\Rightarrow$ Let us consider pressure $p = 0$ as the property to be monitored for this unsteady state problem.

As $p = 0$ is the property,

definitely $\frac{Dp}{Dt} = 0$

$$\frac{Dp}{Dt} = 0 = \frac{\partial p}{\partial t} + (\vec{V}_p \cdot \nabla) p \quad \rightarrow (1)$$

where $\vec{V}_p$ is the velocity with which the interface where $p = 0$ moves (up or down) due to accretion



The flux that crosses the phreatic surface can be given as:

$$(\vec{q} - \vec{N}) \cdot \hat{n} = n_e \vec{V}_p \cdot \hat{n}$$

i.e. $\vec{V}_p = \frac{1}{n_e}(\vec{q} - \vec{N})$

$\therefore$ Equation (1) becomes:

$$\frac{\partial \rho}{\partial t} + \frac{1}{n_e}\left(\vec{q} - \vec{N}\right) \cdot \nabla \rho = 0$$

Expanding the term and incorporating relations of ρ and ϕ for incompressible fluid, we will get the boundary condition equation.