LECTURE 27

21 - MAR - 2014

INITIAL AND BOUNDARY VALUE PROBLEMS

Till now we discussed about the flow and transport processes in porous media by utilising fundamental concepts.

* We developed mathematical expressions for conservations of mass and momenta especially in the form of partial differential equations (pde).
* One need to solve for the dependent variables in pde s to get a better modeling picture of the flow and transport phenomenon in porous media.
* You may be able to interpret infinite solutions for any partial differential equation if no conditions are specified. (That is we can assume various scenarios and conditions and obtain different solutions).
* However, the pde's that are developed in this course are applicable, if you specify appropriate
  - → Initial conditions
  - → Boundary conditions

Those of you who have studied numerical methods may know the meaning of initial conditions and boundary conditions.

* The independent variables utilised mostly in our pde s are → x, y, z, t

* The dependent variables are → piezometric head (ϕ), pressure head (p), liquid saturation (S_l), gas saturation (S_g), water content (θ), specific discharge (q), hydraulic conductivity (K), solute concentrations (c), etc.

* As discussed earlier the partial differential equations are mathematical models of actual phenomenon

  Therefore, by applying appropriate I.C. and B.C.s we will be able to interpret the influences of various variables on transport process.

⇒ Initial Conditions:

In time dependent problems, we may need to specify the initial state or condition of the system. That is we may need to specify the values of dependent variable throughout the domain at time $t = 0$ (Initial Condition)

⇒ Boundary Condition :

As the pdes used in groundwater hydrology (or porous media) are mostly time dependent and spatially varying problems, we may require to provide in addition to initial conditions, the boundary conditions to solve the pdes.

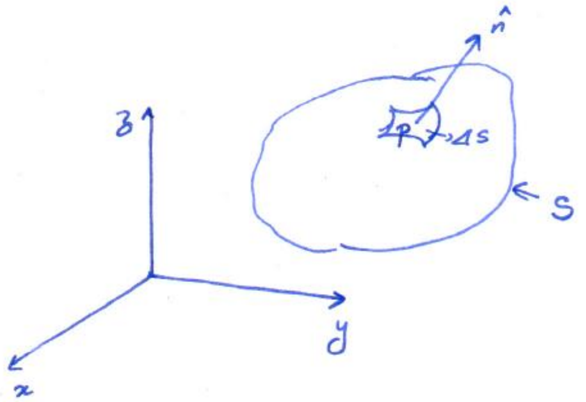
The B.C.s could be based on:

→ geometery of the domain
→ interaction of the system with outside of domain, etc.

We may specify the actual values of dependent variable, or properties of dependent variable at some locations in the domain.

→ For second-order pdes in general we need to specify B.C.'s
↳ the geometeric shape of the boundary
↳ How the dependent variable and its derivatives vary on the boundary.

⇒ For simplicity - we may always take the co-ordinate in principal directions for the anisotropic porous medium.



Consider a three - dimensional domain of anisotropic porous medium in Cartesian coordinate system. The domain is bounded by fixed boundary surface S.

The equation of the boundary surface is:

$F(x, y, z) = 0$ (To describe mathematically a surface)

($\because$ A surface has a common property and we can interpret such an equation).

In two- dimensional system: $F(x, y) = 0$

→ For any mathematical point P on this surface, let $\hat{n}$ be the unit outward normal.

i.e. $\hat{n} = n_x \hat{i} + n_y \hat{j} + n_z \hat{k}$

$$\therefore \quad \nabla F = \frac{\partial F}{\partial x} \hat{i} + \frac{\partial F}{\partial y} \hat{j} + \frac{\partial F}{\partial z} \hat{k}$$

Now $$\hat{n} = \frac{\nabla F}{|\nabla F|} = n_x \hat{i} + n_y \hat{j} + n_z \hat{k}$$

$$\therefore \quad n_x = \frac{1}{|\nabla F|} \frac{\partial F}{\partial x}, \quad n_y = \frac{1}{|\nabla F|} \frac{\partial F}{\partial y}$$

and $$n_z = \frac{1}{|\nabla F|} \frac{\partial F}{\partial z}$$

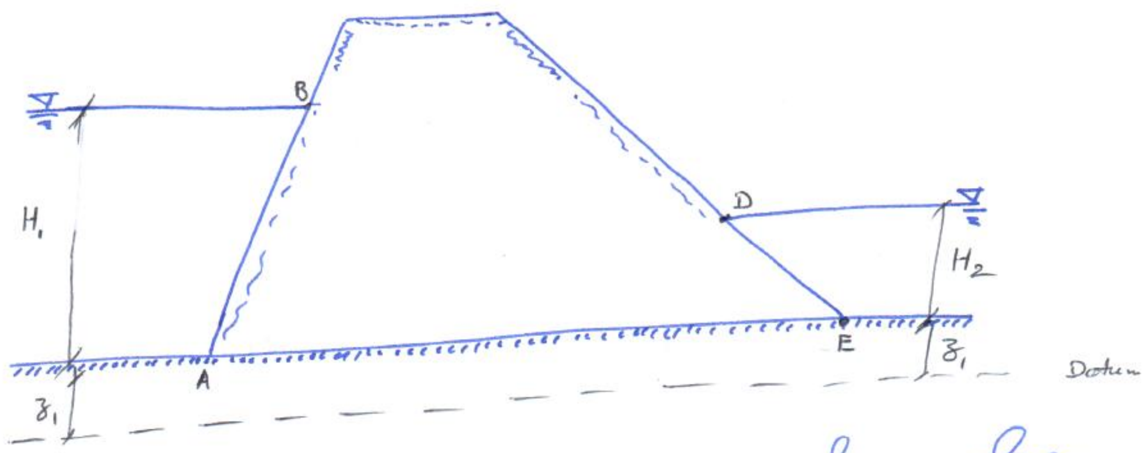
$$(\nabla F)^2 = \left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2 + \left(\frac{\partial F}{\partial z}\right)^2$$

1) Boundary of Prescribed Potential

Recall the piezometric head $\phi(x, y, z)$. It is the gradient of piezometric head that causes ground water flow.
So more or less, we can assume piezometric head as a potential.
We have on the surface now

$$\phi = \phi(x, y, z, t)$$

or $\phi = \phi(x, y)$ in 2D etc.



An earthern bund cross section is shown above.
On the left, the height of water surface is H_1 and on the right it is H_2.
$\Rightarrow$ The earthern bund is a porous media and there will be flow of water from left to right.

Q: What will be the boundary conditions on the left and right ??

Ans: We want to find about B.C's along AB and DE.

The total energy head on water body on left

$$E_1 = H_1 + z_1 + \frac{V_1^2}{2g}$$

However, the water is more or less stagnant and therefore $\frac{V_1^2}{2g} \approx 0.0$.

$$\therefore \quad E_1 = H_1 + z_1 = \phi_1$$

||ly Energy head on right:

$$E_2 = H_2 + z_1 = \phi_2$$

If we go along line AB, the piezometric head at A

$$= H_1 + z_1 = \phi_1$$

At B,

$$= 0 + z_1 + H_1 = \phi_1$$

i.e along line AB:, the piezometric head is constant $= \phi_1$

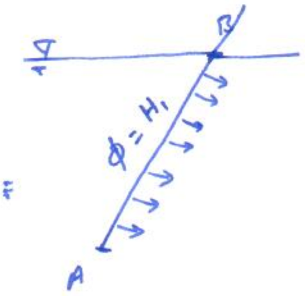
||ly along line DE:, the piezometric head is constant $= \phi_2$

These piezometric heads along AB and DE will be the boundary conditions.

AB $\rightarrow$ equipotential boundary surface

DE $\rightarrow$ equipotential boundary surface.

Such type of boundaries are Dirichlet boundary conditions.
(Please note that equipotential surface does not mean that there will be no flow across the surface).



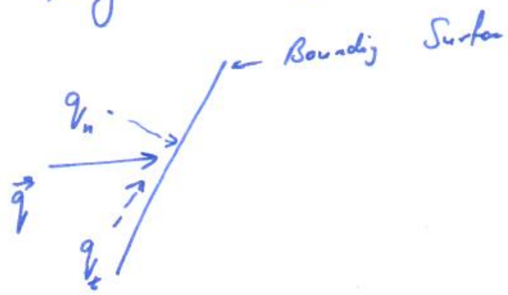
Hydraulic gradient $\nabla\phi$ will be perpendicular to the surface AB.

2) Boundary of Prescribed flux

For such boundary surfaces, flux normal to the surface is prescribed for all points of the boundary

i.e. If $\vec{q}$ is Darcy's specific discharge

Then along the surface, the component of $\vec{q}$ normal to ~~surface is~~ surface is q_n is specified.



i.e. $q_n = \vec{q} \cdot \hat{n}$

⇒ For isotropic porous medium we may specify:

$$\nabla\phi \cdot \hat{n} = \frac{\partial \phi}{\partial n} \quad \text{on surface } S.$$

⇒ In impervious boundary, we have

$$\vec{q} \cdot \hat{n} = 0$$

Recall $\hat{n} = \dfrac{\nabla F}{|\nabla F|}$

$$\therefore \vec{q} \cdot \hat{n} = 0 = \left(-K_x \frac{\partial \phi}{\partial x} \cdot \frac{\partial F}{\partial x}\right) + \left(-K_y \frac{\partial \phi}{\partial y} \frac{\partial F}{\partial y}\right) + \left(-K_z \frac{\partial \phi}{\partial z} \frac{\partial F}{\partial z}\right) = 0$$

i.e. $$K_x \frac{\partial \phi}{\partial x} \frac{\partial F}{\partial x} + K_y \frac{\partial \phi}{\partial y} \frac{\partial F}{\partial y} + K_z \frac{\partial \phi}{\partial z} \frac{\partial F}{\partial z} = 0$$

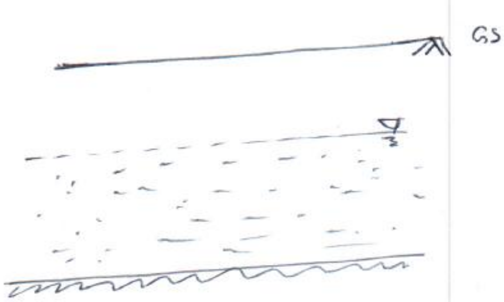
will be the boundary condition.

This is Neuman's boundary condition.

⇒ Mixed boundary conditions (Cauchy's B.C.'s) are specified for cases where we need to specify both the derivative and the dependent variable value at the boundaries.

3) Phreatic Surface without Accretion

Phreatic surface is also called water table.
This is an interfacial surface.
Pressure is usually atmospheric.



⇒ In groundwater hydrology, the phreatic surface is very important.

⇒ At present, we don't know the location and geometric shape of this phreatic surface.
(How to identify) → these are also not fixed boundaries)

If we want to take water table or phreatic surface as a boundary in groundwater flow studies, then we need to accept that the water table is a floating boundary. We cannot incorporate fixed boundary conditions along this surface.

$\Rightarrow$ An important aspect on water table is liquid pressure $p = 0$.

$$\therefore \phi = \frac{p}{\rho g} + z = 0 + z = \underline{\underline{z}}$$

$\therefore$ Equation for water table will be

$$\phi(x, y, z) - z = 0$$

So the earlier described criteria $F(x, y, z) = 0$ is here

$$\underline{\underline{\phi(x, y, z) - z = 0}} = F(x, y, z)$$

If the problem is time dependent, then

$$\phi(x, y, z, t) - z = F(x, y, z, t) = 0$$

along the water table surface S.

$$\therefore \nabla F = \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \left(\frac{\partial \phi}{\partial z} - 1\right)\hat{k} = 0$$

There will not be any flow across the water ~~surf~~ table.

i.e. $\vec{q} \cdot \hat{n} = 0$.

i.e. $\hat{n} = \dfrac{\nabla F}{|\nabla F|}$ and $\vec{q} = -\underline{\underline{K}} \cdot \nabla \phi$

$$\therefore \vec{q} \cdot \hat{n} = \left(-K_x \frac{\partial \phi}{\partial x} \cdot \frac{\partial \phi}{\partial x}\right) + \left(-K_y \frac{\partial \phi}{\partial y} \frac{\partial \phi}{\partial y}\right) + \left(-K_z \frac{\partial \phi}{\partial z} \cdot \frac{\partial \phi}{\partial z}\right) - \left(-K_z \frac{\partial \phi}{\partial z}\right) = 0$$

i.e. $$\boxed{K_x\left(\frac{\partial \phi}{\partial x}\right)^2 + K_y\left(\frac{\partial \phi}{\partial y}\right)^2 + K_z\left(\frac{\partial \phi}{\partial z}\right)^2 - K_z \frac{\partial \phi}{\partial z} = 0}$$

This is the boundary condition along a steady phreatic surface without accretion in an anisotropic medium.

For isotropic medium:

$$\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2 + \left(\frac{\partial \phi}{\partial z}\right)^2 - \frac{\partial \phi}{\partial z} = 0$$

As there is no flow across water table, it automatically becomes a streamline

Ψ = constant

where Ψ = stream function.