LECTURE 26

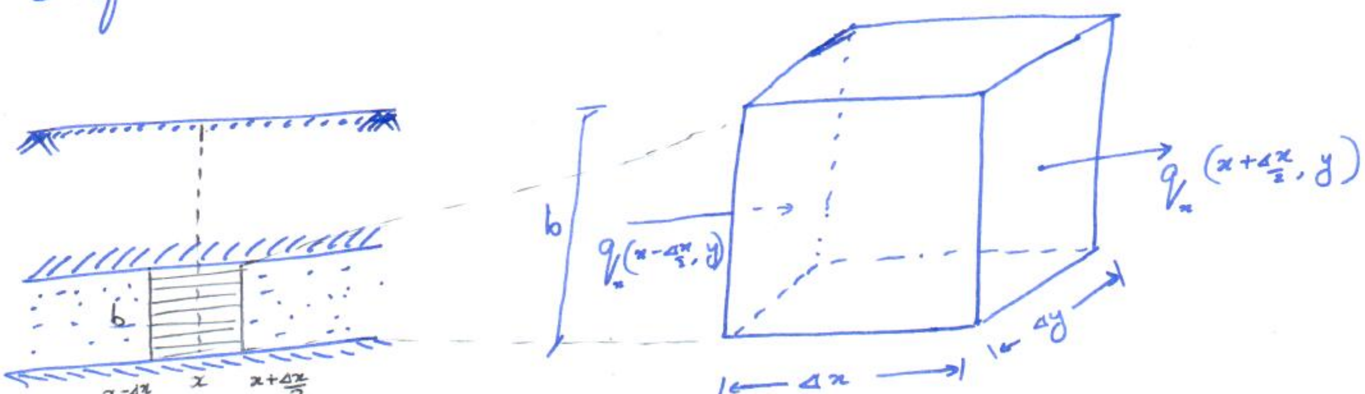
20-MARCH-2014

# CONTINUITY EQUATION IN CONFINED AND LEAKY AQUIFERS

In the last class we started discussing about the continuity in aquifers.

In groundwater hydrology, we generally assume water is incompressible.

Therefore in a control volume from a confined aquifer




We used the Reynolds Transport theorem to develop the continuity equation for the above control volume of confined aquifer of size $\Delta x$, $\Delta y$, and **b**.

If $B$ = extensive property = mass of water in the CV

$\beta$ = intensive property = 1

Then $$\frac{DB}{Dt} = 0 = \frac{\partial}{\partial t}\iiint \beta\rho \, dU + \iint_{cs} \beta\rho(\vec{V}\cdot\hat{n})\, dA$$

i.e. $$0 = \frac{\partial}{\partial t}\iiint \rho \, dU + \iint_{cs} \rho(\vec{q}\cdot\hat{n})\, dA$$

Due to incompressibility: $$0 = \frac{\partial}{\partial t}\iiint dU + \iint_{cs}(\vec{q}\cdot\hat{n})\, dA$$

As per the assumption, the flow in confined aquifer is horizontal. Therefore, the control volume above will have fluxes only in $x$-, and $y$-directions.

$\Rightarrow$ As per the given question: we will now have:

$$\iint_{cs} (\vec{q} \cdot \hat{n})\, dA \approx \frac{\partial}{\partial x}(q_x)\, \Delta x \Delta y\, b + \frac{\partial}{\partial y}(q_y)\, \Delta y \Delta x\, b$$

$$= \Delta x \Delta y\, b \left[ \frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} \right]$$

Recall about the terms $\rightarrow$ aquifer storativity and specific storativity

$\therefore$ The first term of RHS of RTT

$$\frac{\partial}{\partial t} \iiint_{cv} dU_w = ??$$

Aquifer storativity, $S = \dfrac{\Delta U_w}{\Delta A\, \Delta \phi}$

or $\Delta U_w = S\, \Delta A\, \Delta \phi$

$\therefore$ ~~We c~~

$\therefore$ We can say $\dfrac{\partial}{\partial t} \iiint_{cv} dU_w \approx S\, \Delta x \Delta y \cdot \dfrac{\Delta \phi}{\Delta t}$

$$= S\, \Delta x\, \Delta y\, \frac{\left[\phi(t+\Delta t) - \phi(t)\right]}{\Delta t}$$

$\therefore$ RTT becomes:

$$0 = S\, \frac{\Delta x\, \Delta y}{\Delta t} \left[ \phi(t+\Delta t) - \phi(t) \right] + \Delta x\, \Delta y\, b \left[ \frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} \right]$$

i.e. $$-\frac{\partial q_x}{\partial x} - \frac{\partial q_y}{\partial y} = \frac{S}{b\, \Delta t} \left[ \phi(t+\Delta t) - \phi(t) \right]$$

or

if $\Delta x \to 0$, $\Delta y \to 0$, and $\Delta t \to 0$, we have:

$$\boxed{-\frac{\partial q_x}{\partial x} - \frac{\partial q_y}{\partial y} = \frac{S}{b}\frac{\partial \phi}{\partial t}} \longrightarrow (1)$$

Recall, $\vec{q} = -\underline{\underline{K}} \cdot \nabla \phi$

or $q_i = -K_{ij} \dfrac{\partial \phi}{\partial x_j}$

However for confined aquifer

Transmissivity, $\underline{\underline{T}} = \underline{\underline{K}}\, b$ || In principal directions $T_x$, $T_y$, & $T_z$

or $T_{ij} = K_{ij}\, b$

$\therefore$ Equation (1) can be represented now as:

$$-\left[\frac{\partial}{\partial x}\left(-K_x \frac{\partial \phi}{\partial x}\right) b + \frac{\partial}{\partial y}\left(-K_y \frac{\partial \phi}{\partial y}\right) b\right] = S\frac{\partial \phi}{\partial t}$$

or $$-\left[-\frac{\partial}{\partial x}\left(T_x \frac{\partial \phi}{\partial x}\right) - \frac{\partial}{\partial y}\left(T_y \frac{\partial \phi}{\partial y}\right)\right] = S\frac{\partial \phi}{\partial t}$$

or $$\boxed{\frac{\partial}{\partial x}\left(T_x \frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(T_y \frac{\partial \phi}{\partial y}\right) = S\frac{\partial \phi}{\partial t}} \longrightarrow (2)$$

We can also represent in terms of hydraulic conductivity:

$$\frac{\partial}{\partial x}\left(K_x \frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(K_y \frac{\partial \phi}{\partial y}\right) = \frac{S}{b}\frac{\partial \phi}{\partial t}$$

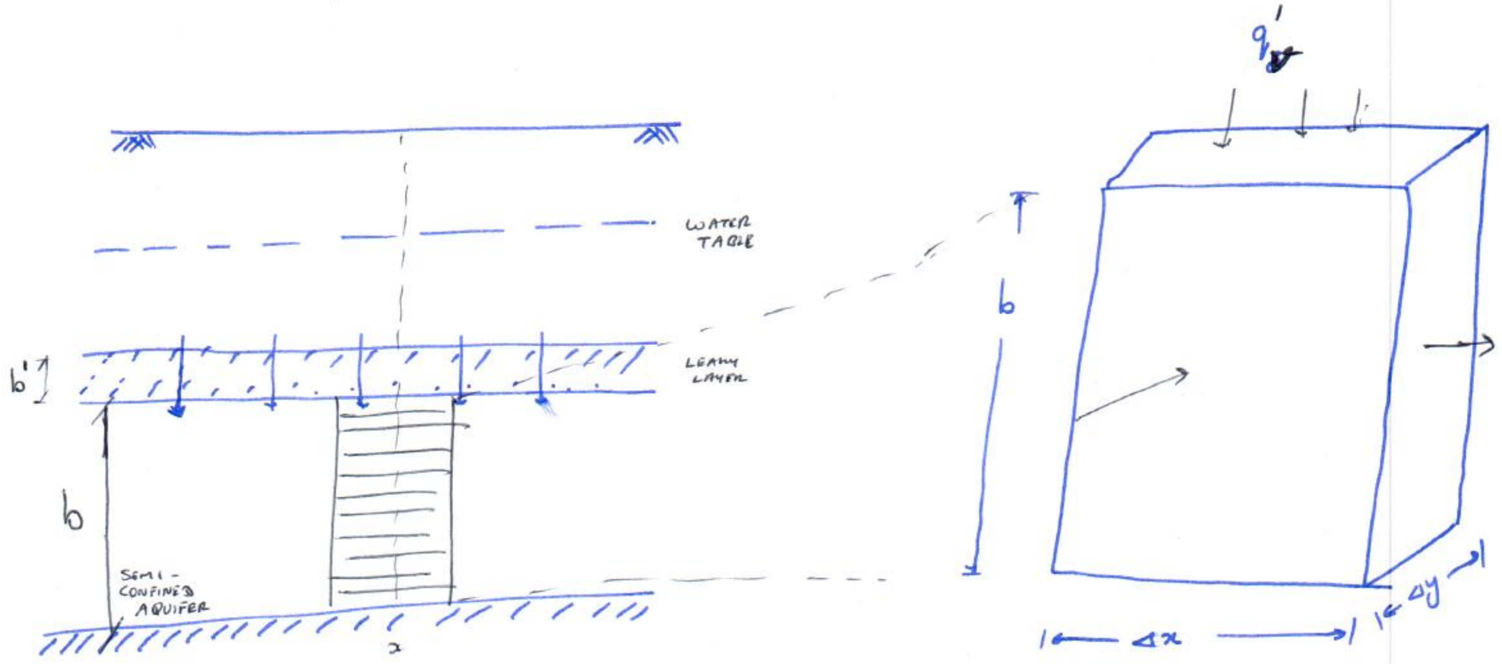
|| $\underline{\underline{K}} = \begin{pmatrix} K_{11} & 0 & 0 \\ 0 & K_{22} & 0 \\ 0 & 0 & K_{33} \end{pmatrix}$

or in index notation:

$$\boxed{\frac{\partial}{\partial x_i}\left(K_{ij} \frac{\partial \phi}{\partial x_j}\right) = S_s \frac{\partial \phi}{\partial t}} \longrightarrow (3)$$

|| $S_s \to$ specific storativity

# Continuity in Leaky Aquifer




Here also we will consider a control volume as shown above for the aquifer.

* The flow in the aquifer is horizontal
* Flow in the leaky semi-pervious layer is essentially vertical
* Water is incompressible

As the flow in the leaky layer is essentially vertical:

$$q'_v = -K' \frac{\partial \phi'}{\partial z}$$

where $K'$ → hydraulic conductivity of leaky layer in vertical direction

$\phi'$ → change in piezometric head in the leaky layer that cause flow in vertical direction.

To apply RTT to develop continuity equation :

$$\frac{\partial}{\partial t}\iiint_{cv} dU_w \quad + \quad \iint_{cs} (\vec{q}\cdot\hat{n})\, dA \quad = 0$$

(Recall Incompressible liquid)

Now

$$\iint_{cs} (\vec{q}\cdot\hat{n})\, dA \quad = \quad \Delta x\, \Delta y\; b\left[\frac{\partial q_{x}}{\partial x} + \frac{\partial q_{y}}{\partial y}\right] - q_v'\, \Delta x \Delta y$$

If $\bar{\phi}$ is the ~~average~~ vertically averaged piezometric head - at any location in aquifer.

i.e. $\cancel{\bar{\phi} = \frac{1}{b}}$ $\quad \bar{\phi}(x, y, t) = \frac{1}{b}\int_{z=0}^{b} \phi(x, y, z, t)\, dz$

We have

$$\frac{\partial}{\partial t}\iiint_{cv} dU_w \quad \approx \quad S\, \Delta x\, \Delta y\; \frac{\bar{\phi}(t+\Delta t) - \bar{\phi}(t)}{\Delta t}$$

∴ In RTT :

$$S\, \Delta x\, \Delta y\; \frac{\bar{\phi}(t+\Delta t) - \bar{\phi}(t)}{\Delta t} \quad + \quad \Delta x \Delta y\, b\left[\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y}\right]$$

$$- q_v'\, \Delta x\, \Delta y \quad = \quad 0$$

i.e. $$\frac{\partial q_{x}}{\partial x} + \frac{\partial q_{y}}{\partial y} - \frac{q_v'}{b} = -\frac{S}{b}\,\frac{\left[\bar{\phi}(t+\Delta t) - \bar{\phi}(t)\right]}{\Delta t}$$

If $\Delta t \to 0$ then :

$$\boxed{\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} - \frac{q_v'}{b} = -\frac{S}{b}\,\frac{\partial \bar{\phi}}{\partial t}} \quad \to \quad (4)$$

Equation (4) can be written in terms of hydraulic conductivity or transmissivity :

i.e. $q_x = -K_x \frac{\partial \bar{\phi}}{\partial x}$ , $q_y = -K_y \frac{\partial \bar{\phi}}{\partial y}$

i.e. $$\frac{\partial}{\partial x}\left(K_x \frac{\partial \bar{\phi}}{\partial x}\right) + \frac{\partial}{\partial y}\left(K_y \frac{\partial \bar{\phi}}{\partial y}\right) + \frac{q_v'}{b} = S_s \frac{\partial \bar{\phi}}{\partial t}$$

If $T_{ij} = K_{ij}\, b$

Then :

$$\boxed{\frac{\partial}{\partial x_i}\left(T_{ij} \frac{\partial \bar{\phi}}{\partial x_j}\right) - K' \frac{\partial \phi'}{\partial z} = S \frac{\partial \bar{\phi}}{\partial t}} \rightarrow (5)$$

⇒ If there is overlying water table aquifer over the leaky semi-pervious layer, then if $\bar{\phi}_0$ is the average piezometric head in that water table aquifer

You have $$\frac{\partial \phi'}{\partial z} \approx \frac{\bar{\phi}_0 - \bar{\phi}}{b'}$$

$$\therefore\ q_v' = -K' \frac{\bar{\phi}_0 - \bar{\phi}}{b'}$$

Please Note: : for isotropic homogeneous porous medium,
$T$ = constant , $K$ = constant

$$\therefore\ \frac{\partial^2 \bar{\phi}}{\partial x^2} + \frac{\partial^2 \bar{\phi}}{\partial y^2} - \frac{K'}{K b' b}(\bar{\phi}_0 - \bar{\phi}) = \frac{S}{T} \frac{\partial \bar{\phi}}{\partial t}$$

i.e. $$\frac{\partial^2 \bar{\phi}}{\partial x^2} + \frac{\partial^2 \bar{\phi}}{\partial y^2} - \frac{K'}{T b'}(\bar{\phi}_0 - \bar{\phi}) = \frac{S}{T} \frac{\partial \bar{\phi}}{\partial t} \rightarrow (6)$$

## Pathlines, Streamlines, etc.

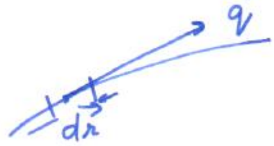
From your fluid mechanics classes, recall the terms:

* Pathlines
* Stream functions & Streamlines
* Equi-potential functions & Equi-potential lines, etc.

Pathline $\rightarrow$ Lagrangian concept, time dependent
Streamline $\rightarrow$ Instantaneous lines

In porous media flows, we can have streamlines:

$$\vec{q} = q_x \hat{i} + q_y \hat{j}$$




$$\vec{q} \times \vec{dr} = 0$$

This is the streamline equation

In two dimensional case:

$$\frac{dx}{q_x} = \frac{dy}{q_y}$$

or

$$\frac{dx}{K_x \frac{\partial \phi}{\partial x}} = \frac{dy}{K_y \frac{\partial \phi}{\partial y}}$$

For isotropic porous medium:

$$\frac{dx}{\partial \phi / \partial x} = \frac{dy}{\partial \phi / \partial y}$$

i.e.

$$\frac{\partial \phi}{\partial y} dx - \frac{\partial \phi}{\partial x} dy = 0$$

or

$$\boxed{\nabla \phi \times \vec{dr} = 0}$$

~~Solution of~~ From streamline criteria:

$$\frac{dx}{q_x} = \frac{dy}{q_y}$$

We have: $q_y\,dx - q_x\,dy = 0 \rightarrow$ (A)

This is a differential equation.

Solution of (A) is $\psi(x,y) = \text{constant}$

$\psi \rightarrow$ stream function

such that

$$d\psi = \frac{\partial \psi}{\partial x}dx + \frac{\partial \psi}{\partial y}dy = 0$$

For incompressible liquid $\nabla \cdot \vec{q} = 0$

$$\therefore q_x = -\frac{\partial \psi}{\partial y} \quad \text{and} \quad q_y = \frac{\partial \psi}{\partial x}.$$

Ily we can also define equi-potential function.