LECTURE 25

14-MAR-2014

CONTINUITY OF THREE PHASES IN A THREE-DIMENSIONALLY CONSOLIDATING POROUS MEDIA

In the last class, we started discussing about the mass conservation equations for three phases

→ solids, liquids, and gases

in a consolidating porous media

Recall in the control volume of porous media:

Density of solids $= \rho_s$

Density of liquid $= \rho_l$

Density of gas $= \rho_g$

It is assumed that mass of solids in the control volume will not change, although the solids move at velocity $\vec{V_s}$.

Mass flux for solid phase, $\vec{J_s} = \rho_s \vec{q_s}$

" " for liquid phase, $\vec{J_l} = \rho_l \vec{q_l}$

" " for gaseous phase, $\vec{J_g} = \rho_g \vec{q_g}$

Recall the mass conservation for liquid in saturated porous media: $\nabla \cdot (\rho \vec{q}) + \frac{\partial}{\partial t}(n \rho) = 0$

Similarly we can have mass conservation equation for each phase as below:

$$\left.\begin{aligned}
\text{Liquids:}\quad & \nabla \cdot \vec{J_\ell} + \frac{\partial}{\partial t}\left(n\, S_\ell\, \rho_\ell\right) = 0 \\
\text{Gas:}\quad & \nabla \cdot \vec{J_g} + \frac{\partial}{\partial t}\left((1 - S_\ell)\, n\, \rho_g\right) = 0 \\
\text{Solids:}\quad & \nabla \cdot \vec{J_s} + \frac{\partial}{\partial t}\left((1 - n)\, \rho_s\right) = 0
\end{aligned}\right\} \rightarrow (1)$$

Recall in the earlier section, we described the relative specific discharge of liquid in the saturated porous media, where the solids were moving at a velocity $\vec{V_s}$.

Similarly we need to describe the specific discharge of liquid w.r.t. the moving solids here also

i.e. $\vec{q_{r\ell}} = -K\, \nabla \phi^*$

(Note that the porous media is isotropic & homogeneous)

i.e. $\vec{q_{r\ell}} = (\vec{V_\ell} - \vec{V_s})\, n\, S_\ell$

where $\vec{V_\ell} \rightarrow$ actual velocity of liquid w.r.t. fixed coordinates
$\vec{V_s} \rightarrow$ velocity of solids in the control volume.

$\phi^* \rightarrow$ piezometric head for compressible fluid.
$\rho_\ell \rightarrow \rho_\ell(p)$ (i.e. function of liquid pressure)

$\Rightarrow$ The equations of state for the three phases ~~are~~ that may be followed are:

$$\left.\begin{aligned}
\rho_s &= \text{constant} \\
\rho_\ell &= \rho_{0\ell}\, \exp(\beta p) \\
\rho_g &= \rho_{0g}\left(\frac{p}{p_{0g}}\right)
\end{aligned}\right\} \rightarrow (2)$$

As discussed earlier, $\rho_{0\ell}$, ρ_{0g}, etc. are reference densities.

Utilising these relations in mass conservation equations 1:

For solids:

$$\nabla \cdot \vec{J_s} + \frac{\partial}{\partial t}\left((1-n)\rho_s\right) = 0$$

i.e. $$\nabla \cdot \left(\rho_s \vec{q_{V_s}}\right) + \frac{\partial}{\partial t}\left((1-n)\rho_s\right) = 0$$

As $\rho_s \approx$ a constant $\nabla \rho_s = 0$ and $\frac{\partial}{\partial t}\rho_s = 0$

∴ We have:

$$\nabla \cdot \vec{q_{V_s}} + \frac{\partial}{\partial t}(1-n) = 0$$

or $$\frac{\partial n}{\partial t} = (1-n)\,\nabla \cdot \vec{V_s} + \vec{V_s} \cdot \nabla(1-n)$$

∵ $$\vec{q_{V_s}} = (1-n)\vec{V_s}$$

∴ $$\frac{\partial n}{\partial t} = \nabla \cdot \left[(1-n)\vec{V_s}\right] \longrightarrow 3$$

For liquids:

$$\nabla \cdot \vec{J_\ell} + \frac{\partial}{\partial t}\left(n S_\ell \rho_\ell\right) = 0$$

or $$\vec{J_\ell} = \rho_\ell \vec{q_{V_\ell}} = \rho_\ell\, n\, S_\ell \vec{V_\ell}$$

∴ $$\nabla \cdot \left[\rho_\ell\, n\, S_\ell \vec{V_\ell}\right] + \frac{\partial}{\partial t}\left(n S_\ell \rho_\ell\right) = 0 \rightarrow 4$$

For gas :

$$\nabla \cdot \vec{J}_g + \frac{\partial}{\partial t}\left((1 - S_\ell)\, n\, \rho_g \right) = 0$$

i.e. $$\nabla \cdot \left[(1 - S_\ell)\, n\, \rho_g\, \vec{V}_g \right] + \frac{\partial}{\partial t}\left(n \rho_g (1 - S_\ell) \right) = 0 \quad \rightarrow (5)$$

Equations (3), (4), and (5) are to be solved simultaneously for such multi-phase systems.

CONTINUITY EQUATION FOR FLOW IN CONFINED AND LEAKY AQUIFERS

In groundwater hydrology you will often come up with terms like - aquifers.

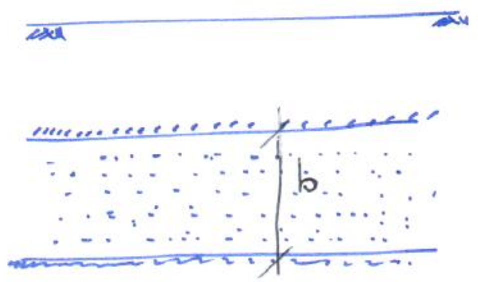
You have confined, unconfined, and leaky aquifers in nature.

⇒ The theory of mass conservation developed in general for any porous media are very much applicable for mass conservation in aquifers.

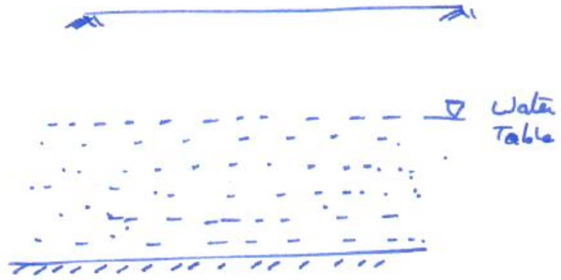
→ Let us revisit the mass conservation equations for aquifers in this portion.

⇒ An aquifer is a water bearing strata in subsurface of earth.

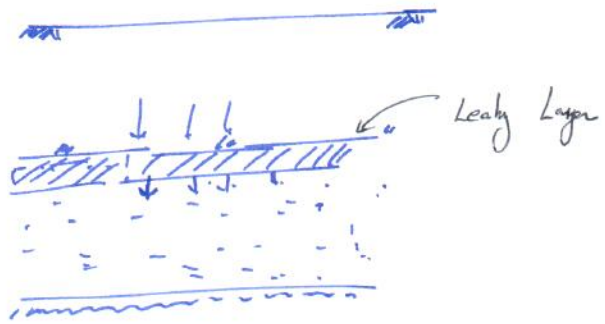
A confined aquifer



An unconfined aquifer



A Leaky aquifer



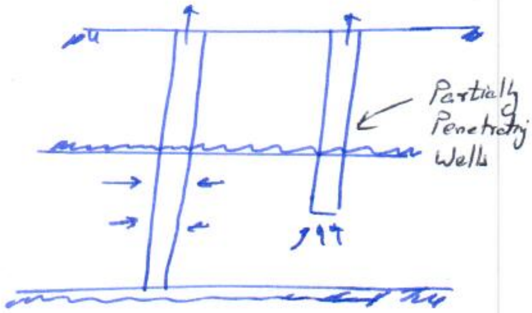
→ Compared to flow in general in porous media, aquifer flows are <u>typically horizontal in nature</u>.

→ We may also approximate flow in aquifers as horizontal.

→ The significance of horizontal flow approximations in aquifers are that

* For confined aquifers the flow is always in x-y plane (No Vertical flow)
* In leaky aquifers, although there are some quantities that fill the aquifer from top layers, still the basic flow is horizontal. The vertical flow will be considered only as a source or sink to the control volume.

→ However this approximation may not be fully true for partially penetrating wells



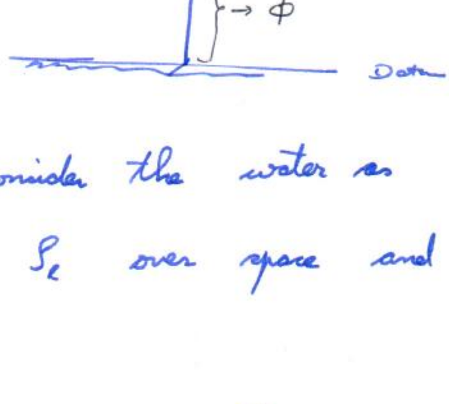
⇒ For aquifer flows:

The piezometric head $\phi = \phi(x, y, z, t)$ in actual three-dimensional flows:

However two-dimensional flows may be considered:

$$\bar{\phi} = \bar{\phi}(x, y, t)$$

where $\bar{\phi}$ is the average of piezometric head taken along a vertical line.



⇒ In groundwater aquifers, we consider the water as <u>incompressible</u>. ∴ Variations in ρ_e over space and time are neglected.

⇒ For a homogeneous isotropic aquifer, we have

Transmissivity, $T = K b$

where K → hydraulic conductivity

b → confined aquifer thickness.

In Confined Aquifers

⇒ Transmissivity , $T = \bar{K} b$

$\bar{K}$ → average hydraulic conductivity over a vertical line.

$(\because \; K = K(x, y))$

$$\bar{K} = \frac{1}{b} \int_0^b K(z)\, dz$$

⇒ Aquifer Storativity :

Volume of water (ΔU_w) released from aquifer storage per unit horizontal area of aquifer and per unit decline of the average piezometric head in the aquifer

$$S = \frac{\Delta U_w}{\Delta A \, \Delta \bar{\Phi}}$$

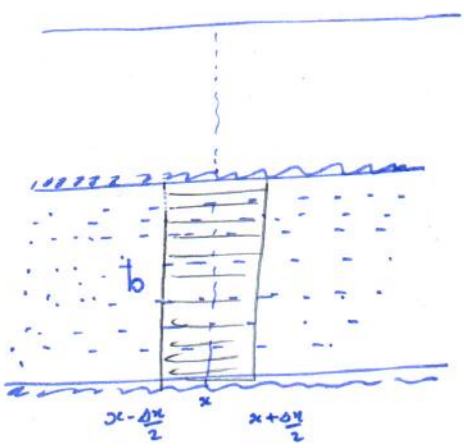
⇒ Specific Storativity :

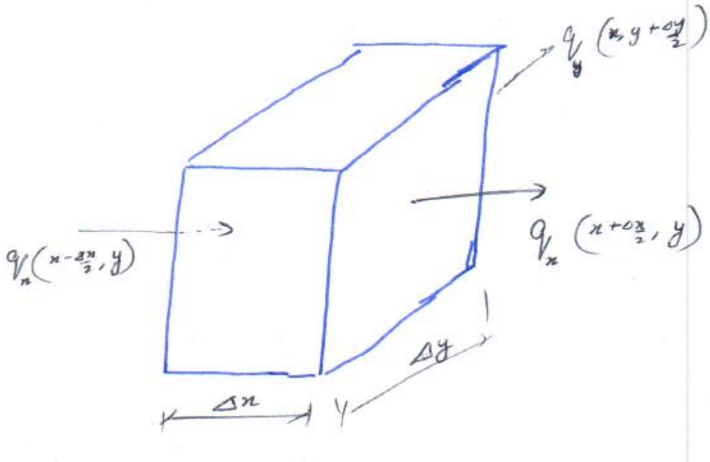
Volume of water released from aquifer storage per unit bulk volume of aquifer per unit decline of piezometric head. (S_s).

$$S = S_s b$$

⇒ To develop the continuity equation :

Let us consider the confined aquifer of thickness b and let us use the control volume approach:





The Control Volume

Mass Inflow in the x- direction $= \rho\, q_x (x - \frac{\Delta x}{2}, y) \,.\, b \,.\, \Delta y$

Mass Outflow in the x- direction $= \rho\, q_x (x + \frac{\Delta x}{2}, y) \,.\, b \,.\, \Delta y$

$\therefore$ Net mass outflow in the x- direction $= \rho \left[q_x (x + \frac{\Delta x}{2}, y) - q_x (x - \frac{\Delta x}{2}, y) \right] \Delta y\, b$

||ly Net mass outflow in the y- direction $= \rho \left[q_y (x, y + \frac{\Delta y}{2}) - q_y (x, y - \frac{\Delta y}{2}) \right] \Delta x\, b$

Reynolds Transport Theorem for mass conservation

$$\frac{DB}{Dt} = 0 = \frac{\partial}{\partial t} \iiint \beta \rho \, dV + \iint_{cs} \beta \rho \, (\vec{V} \cdot \hat{n})\, dA$$

Here $B =$ mass of water and $\frac{DB}{Dt} = 0$, $\beta = 1.0$

$$\therefore 0 = \frac{\partial}{\partial t} \iiint \rho \, dV + \iint_{cs} \rho (\vec{q} \cdot \hat{n})\, dA$$ for porous CV.

As liquid is incompressible ρ is cancelled.

$$0 = \frac{\partial}{\partial t} \iiint dV + \iint_{cs} (\vec{q} \cdot \hat{n})\, dA \rightarrow (6)$$