LECTURE 24
13-MAR-2014

The Continuity Equation Of A Homogeneous Compressible fluid in a deforming porous media

In the last class, we arrived at the form

$$\frac{\partial}{\partial x}(\rho q_x) + \frac{\partial}{\partial y}(\rho q_y) + \frac{\partial}{\partial z}(\rho q_z) = -\rho\left[\beta n + \alpha_b'(1-n)\right]\frac{\partial p}{\partial t} \longrightarrow (1)$$

where
$\rho \rightarrow$ density of compressible fluid
$q_x \rightarrow$ Component of Darcy velocity vector in x-direction
$n \rightarrow$ porosity of the media
$p \rightarrow$ pressure in the liquid
$\beta \rightarrow$ liquid compressibility
$\alpha_b' \rightarrow$ Coefficient of soil compressibility $= -\frac{1}{U_b}\frac{\partial U_b}{\partial \sigma'}\Big|_{p=\text{constant}}$

(Please note that, we are assuming that the soil is compressible only in vertical direction).

⇒ Subsequently we suggested that let the porous media that is deforming may be due to the motion of solid grains in the porous media.

→ If the solids are moving at a velocity $\vec{V_s}$

Then specific discharge for liquid, $\vec{q} = \vec{q_r} + n\vec{V_s}$

and the continuity equation for liquids was given as:

$$\nabla\cdot\left[\rho\vec{q_r}\right] + \rho\,\nabla\cdot\vec{V_s} + n\,\vec{V_s}\cdot\nabla\rho + n\frac{\partial \rho}{\partial t} = 0 \longrightarrow (2)$$

From equation (2), for homogeneous incompressible fluid, we have:

$$\nabla \cdot \left[\rho \vec{q}_{V_r} \right] + \nabla \cdot \vec{V_s} = 0$$

Again in equation (2), note that we have taken $\rho = \rho(p)$.

Note $\frac{\partial \rho}{\partial x} = \rho \beta \frac{\partial p}{\partial x}$ and $\frac{\partial \rho}{\partial t} = \rho \beta \frac{\partial p}{\partial t}$, etc.

$\therefore$ Equation (2) becomes:

$$\nabla \cdot \left[\rho \vec{q}_{V_r} \right] + \rho \, \nabla \cdot \vec{V_s} + n \vec{V_s} \cdot \beta \rho \nabla p + n \beta \rho \frac{\partial p}{\partial t} = 0$$

i.e.

$$\nabla \cdot \left[\rho \vec{q}_{V_r} \right] + \rho (\nabla \cdot \vec{V_s}) = - n \beta \rho \left[\frac{\partial p}{\partial t} + (\vec{V_s} \cdot \nabla) p \right]$$

Now defining the term $\left[\frac{\partial}{\partial t} + \vec{V_s} \cdot \nabla \right]$ as the total derivative in Lagrangian with respect to moving solids

i.e. $\frac{D_s}{Dt}(\,)$

We will get:

$$\nabla \cdot \left[\rho \vec{q}_{V_r} \right] + \rho (\nabla \cdot \vec{V_s}) = - n \beta \rho \frac{D_s p}{Dt} \quad \longrightarrow (3)$$

$\Rightarrow$ In fluid transport chapter, if $\vec{V}_{a\alpha}$ is the velocity of any property G fx on α-space, then we can define the term <u>dilatation</u>, such that

$$\nabla \cdot \vec{V}_{a\alpha} = - \delta U \frac{D}{Dt} \left(\frac{1}{\delta U} \right)$$

or

$$\nabla \cdot \vec{V}_{a\alpha} = \frac{1}{\delta U} \frac{D}{Dt} (\delta U)$$

$\delta U \rightarrow$ elementary volume of fluid

$\Rightarrow$ Utilising this concept of dilatation in the bulk porous media, let us define:

$$\nabla \cdot \vec{V_s} = \frac{1}{\delta U_b} \frac{D_s(\delta U_b)}{Dt} \rightarrow (4)$$

As $U_b = U_w + U_s$

and $U_s = (1-n) U_b$

We have $\delta U_s = (1-n)\,\delta U_b$ = constant

(It is assumed that volume of solids remain constant in the control volume).

$$\therefore \frac{D_s}{Dt}(\delta U_s) = 0 = \frac{D_s}{Dt}\left[(1-n)\,\delta U_b\right]$$

$$= (1-n)\frac{D_s}{Dt}(\delta U_b) - \delta U_b \frac{D_s n}{Dt}$$

or $$\frac{1}{\delta U_b}\frac{D_s(\delta U_b)}{Dt} = \frac{1}{1-n}\frac{D_s n}{Dt}$$

$\therefore$ We have now in equation (4):

$$\nabla \cdot \vec{V_s} = \frac{1}{1-n}\frac{D_s n}{Dt} = \frac{1}{\delta U_b}\frac{D_s}{Dt}(\delta U_b)$$

$$= \frac{1}{\delta U_b}\frac{D_s(\delta U_b)}{D\sigma'}\frac{D_s \sigma'}{Dt}$$

$$= -\alpha_b' \frac{D_s \sigma'}{Dt}$$

$$= -\alpha_b'\left[\frac{D_s}{Dt}(\sigma - p)\right]$$

$\therefore$ Now equation (3) becomes:

$$\nabla \cdot \left[\rho \vec{q}_{r}\right] - \rho \alpha_b' \frac{D_s \sigma}{Dt} + \rho \alpha_b' \frac{D_s p}{Dt} = -n\beta\rho \frac{D_s p}{Dt}$$

i.e. $$\nabla \cdot \left[\rho \vec{q}_{r}\right] - \rho \alpha_b' \frac{D_s \sigma}{Dt} = -\rho\left[n\beta + \alpha_b'\right] \frac{D_s p}{Dt} \quad \rightarrow (5)$$

In groundwater hydrology, in most situations, we can assume

$$\frac{D_s \sigma}{Dt} = 0$$

$$\therefore \boxed{\nabla \cdot \left(\rho \vec{q}_{r}\right) = -\rho\left(n\beta + \alpha_b'\right) \frac{D_s p}{Dt}} \quad \rightarrow (6)$$

Again in most topics in groundwater hydrology we can assume $\vec{V_s}$ to be very small. (i.e Velocity of motion of solid particles is small).

Then $$(\vec{V_s} \cdot \nabla) p \ll \frac{\partial p}{\partial t}$$

$$\therefore \frac{D_s p}{Dt} \approx \frac{\partial p}{\partial t}$$

and also $$\vec{q}_{r} \approx \vec{q}$$

Hence the equation (6) becomes:

$$\nabla \cdot \left(\rho \vec{q}\right) = -\rho\left(n\beta + \alpha_b'\right) \frac{\partial p}{\partial t} \quad \rightarrow (7)$$

Recall the expressions used earlier to describe

$$\nabla \cdot \left(\rho \vec{q}\right)$$

$$\nabla \cdot (\rho \vec{q}) = \nabla \cdot [\rho K \nabla \phi^*]$$

where $\phi^* \rightarrow$ piezometric head for compressible fluids.

$$\nabla \rho = \rho \beta \nabla p$$

$$\frac{\partial p}{\partial x} = \rho g \frac{\partial \phi^*}{\partial x}, \quad \frac{\partial p}{\partial y} = \rho g \frac{\partial \phi^*}{\partial y}, \text{ and } \frac{\partial p}{\partial z} = \rho g \left(\frac{\partial \phi^*}{\partial z} - 1 \right)$$

(These are discussed in LECTURE 21)

Substituting all these relations in equation (7), we will get a partial differential equation in terms of ϕ^*

[In new assignment, derive this pde (8), equation no: 8]

$$K \nabla^2 \phi^* - K \beta \rho g \frac{\partial \phi^*}{\partial z} - (\alpha_b' + n\beta) \frac{D_s \phi^*}{Dt} \rho g = 0 \quad \rightarrow (8)$$

Please note that we have considered

$$\frac{D_s \sigma}{Dt} = 0 \quad \text{and} \quad |V_s| <<< |V_r| \quad \therefore \vec{V} \approx \vec{V_r}$$

$$\boxed{K \left[\nabla^2 \phi^* - \beta \rho g \frac{\partial \phi^*}{\partial z} \right] = S_s \rho g \frac{D_s \phi^*}{Dt}} \quad \rightarrow (9)$$

where $S_s \rightarrow$ specific storativity.

Continuity Equation for Three Phases in Porous Media having Three-Dimensional Consolidation

In a porous media if there are three phases

Solids, Gases, Liquids

- eg:

Let us consider the porous media consolidates in all three directions

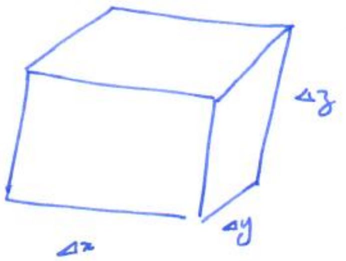
Let

density of solid $= \rho_s$

density of liquid $= \rho_\ell$

density of gas $= \rho_g$

For a fixed control volume in space



Mass fluxes in each phase:

Mass flux for solid phase, $\vec{J}_s = \rho_s \vec{q}_s$

Mass flux for liquid phase, $\vec{J}_\ell = \rho_\ell \vec{q}_\ell$

Mass flux for gaseous phase, $\vec{J}_g = \rho_g \vec{q}_g$

Recall the mass conservation equation for liquid in saturated porous media:

$$\nabla \cdot (\rho \vec{q}) + \frac{\partial}{\partial t}(n\rho) = 0$$

$\Rightarrow$ The porous media porosity $= n$

Control Volume $= \Delta x \, \Delta y \, \Delta z = U_b$

Liquid saturation $= S_\ell \quad ; \quad 0 \le S_\ell \le 1.0$

Gas saturation $= S_g \quad ; \quad 0 \le S_g \le 1.0$

$= 1 - S_\ell$

Volume of solids $= (1-n)\, U_b = (1-n)\, \Delta x \Delta y \Delta z$

Volume of liquids $= n\, S_\ell\, \Delta x \Delta y \Delta z$

Volume of gases $= (1 - S_\ell)\, n\, U_b$

Mass conservation equation for :

Liquids : $\nabla \cdot \vec{J_l} + \frac{\partial}{\partial t}(n S_l \rho_l) = 0$

Gas : $\nabla \cdot \vec{J_g} + \frac{\partial}{\partial t}((1 - S_l) n \rho_g) = 0$

Solids : $\nabla \cdot \vec{J_s} + \frac{\partial}{\partial t}((1 - n) \rho_s) = 0$

→ Specific discharge (using Darcy's law) w.r.t the solids for the liquids (Note: Homogeneous & Isotropic Porous Medium)

$$\vec{q_{lr}} = -K \nabla \phi^*$$

$$\vec{q_{lr}} = (\vec{V_l} - \vec{V_s}) \, n S_l$$

where $\vec{V_l}$ → actual velocity of liquid w.r.t fixed co-ordinates

$\vec{V_s}$ → velocity of solid grains in control volume.

However K varies w.r.t saturation

$$K = K_{sat} K_r \,,$$ where K_r → relative hydraulic conductivity.

$$\text{or} \quad K = \frac{k_{lr} k_{sat} \rho_l g}{\mu}$$

$$\therefore \vec{q_{lr}} = -\frac{k_{lr} k_{sat} \rho_l g}{\mu} \nabla \phi^*$$

k_{sat} → permeability of isotropic homogeneous medium

k_{lr} → relative permeability that depends on saturation

$$k_{lr} = 1.0 \quad \text{for} \quad S_l = 1.0$$
$$= 0.0 \quad \text{for} \quad S_l = 0.0$$

Now $\rho_l = \rho_{0l} \exp(\beta p)$

and $\rho_g = \rho_{og}\left(\frac{p}{p_{og}}\right)$

Also $\vec{q}_r = (\vec{V}_l - \vec{V}_s)\, n\, \rho_l = -\frac{k_{lr} k_{sat} \rho_l g}{\mu} \nabla \Phi^*$

Also note ρ_s = constant :

$\vec{J}_s = \rho_s \vec{q}_s$

Mass conservation : $\nabla \cdot \vec{J}_s + \frac{\partial}{\partial t}\left((1-n)\rho_s\right) = 0$

i.e. $\nabla \cdot \rho_s \vec{q}_s + \frac{\partial}{\partial t}\left((1-n)\rho_s\right) = 0$

i.e. $\nabla \cdot \vec{q}_s + \frac{\partial}{\partial t}(1-n) = 0 \rightarrow (9)$

$\vec{q}_s = (1-n)\vec{V}_s$

$\therefore$ Equation (9) becomes:

$\frac{\partial n}{\partial t} = \nabla \cdot \left[(1-n)\vec{V}_s\right] \rightarrow (10)$