LECTURE 22
07 - MARCH - 2014

CONTINUITY OF COMPRESSIBLE HOMOGENEOUS FLUID IN HOMOGENEOUS ISOTROPIC POROUS MEDIA

Yesterday we arrived or formulated the continuity equation of a compressible homogeneous fluid in an homogeneous isotropic porous media. The fluid density was related to the pressure and the continuity equation was described in terms of fluid pressure.

$$\frac{k}{\mu}\nabla^2 p + \frac{k\beta}{\mu}(\nabla p)^2 + \frac{2k\rho g\beta}{\mu}\frac{\partial p}{\partial z} = n\beta\frac{\partial p}{\partial t} \qquad \rightarrow (1)$$

or

in index notation

$$\frac{k}{\mu}\left[\frac{\partial}{\partial x_i}\left(\frac{\partial p}{\partial x_i}\right) + \beta\frac{\partial p}{\partial x_i}\frac{\partial p}{\partial x_i} + 2\rho g\beta\frac{\partial p}{\partial z}\right] = n\beta\frac{\partial p}{\partial t}$$

$\Rightarrow$ To express continuity equation in terms of piezometric head ϕ^*

Recall

$$\frac{\partial p}{\partial x} = \rho g\frac{\partial \phi^*}{\partial x} \quad ; \quad \frac{\partial p}{\partial y} = \rho g\frac{\partial \phi^*}{\partial y}$$

and $$g\frac{\partial \phi^*}{\partial z} = g + \frac{1}{\rho}\frac{\partial p}{\partial z}$$

and $$\frac{\partial p}{\partial t} = \rho g\frac{\partial \phi^*}{\partial t}$$

Substituting these expressions in the equation

$$k\left[\frac{\partial}{\partial x}\left(\frac{\rho}{\mu}\frac{\partial p}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{\rho}{\mu}\frac{\partial p}{\partial y}\right)+\frac{\partial}{\partial z}\left(\frac{\rho}{\mu}\left(\frac{\partial p}{\partial z}+\rho g\right)\right)\right]=n\frac{\partial \rho}{\partial t} \longrightarrow (2)$$

We will get:

$$k\left[\frac{\partial}{\partial x}\left(\frac{\rho^2 g}{\mu}\frac{\partial \phi^*}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{\rho^2 g}{\mu}\frac{\partial \phi^*}{\partial y}\right)+\frac{\partial}{\partial z}\left(\frac{\rho^2 g}{\mu}\frac{\partial \phi^*}{\partial z}\right)\right]=n\frac{\partial \rho}{\partial t}$$

However, recall $\rho = \rho(p)$ (i.e. $\rho = \rho_0 \exp[\beta(p-p_0)]$)

Also recall, the continuity equation form for this case i.e. eqn (1).

Substitute the relation between $\frac{\partial p}{\partial x}$ and $\frac{\partial \phi^*}{\partial x}$, etc. in eqn (1)

We will get for eqn. (1)

$$\frac{k}{\mu}\left(\frac{\partial^2 p}{\partial x^2}+\frac{\partial^2 p}{\partial y^2}+\frac{\partial^2 p}{\partial z^2}\right)+\frac{\beta k}{\mu}\left(\left(\frac{\partial p}{\partial x}\right)^2+\left(\frac{\partial p}{\partial y}\right)^2+\left(\frac{\partial p}{\partial z}\right)^2\right)$$

$$+\frac{2kg\rho\beta}{\mu}\frac{\partial p}{\partial z} = n\beta\frac{\partial p}{\partial t} \longrightarrow (1)$$

$$\frac{\partial^2 p}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial p}{\partial x}\right) = \frac{\partial}{\partial x}\left(\rho g\frac{\partial \phi^*}{\partial x}\right) = \rho g\frac{\partial^2 \phi^*}{\partial x^2} + g\frac{\partial \rho}{\partial x}\frac{\partial \phi^*}{\partial x}$$

$$= \rho g\frac{\partial^2 \phi^*}{\partial x^2} + g\beta\rho\frac{\partial p}{\partial x}\frac{\partial \phi^*}{\partial x}$$

$$= \rho g\frac{\partial^2 \phi^*}{\partial x^2} + \rho^2 g^2\beta\left(\frac{\partial \phi^*}{\partial x}\right)^2$$

IIly $\frac{\partial^2 p}{\partial y^2} = \rho g\frac{\partial^2 \phi^*}{\partial y^2} + \rho^2 g^2 \beta\left(\frac{\partial \phi^*}{\partial y}\right)^2$

$$\frac{\partial^2 p}{\partial z^2} = \frac{\partial}{\partial z}\left(\frac{\partial p}{\partial z}\right) = \frac{\partial}{\partial z}\left(\rho g\left(\frac{\partial \phi^*}{\partial z} - 1\right)\right)$$

$$= \frac{\partial}{\partial z}\left(\rho g \frac{\partial \phi^*}{\partial z}\right) - \frac{\partial}{\partial z}(\rho g)$$

$$= \rho g \frac{\partial^2 \phi^*}{\partial z^2} + \beta \rho^2 g^2 \left(\frac{\partial \phi^*}{\partial z}\right)^2 - g\beta \rho^2 g \frac{\partial \phi^*}{\partial z} + \beta \rho^2 g^2$$

Again $\left(\frac{\partial p}{\partial x}\right)^2 = \rho^2 g^2 \left(\frac{\partial \phi^*}{\partial x}\right)^2$

$$\left(\frac{\partial p}{\partial y}\right)^2 = \rho^2 g^2 \left(\frac{\partial \phi^*}{\partial y}\right)^2$$

$$\left(\frac{\partial p}{\partial z}\right)^2 = \rho^2 g^2 \left(\frac{\partial \phi^*}{\partial z} - 1\right)^2$$

and $\frac{\partial p}{\partial t} = \rho g \frac{\partial \phi^*}{\partial t}$

∴ We have equation (1) as:

$$\frac{k}{\mu}\rho g\left(\frac{\partial^2 \phi^*}{\partial x^2} + \frac{\partial^2 \phi^*}{\partial y^2} + \frac{\partial^2 \phi^*}{\partial z^2}\right) + \frac{k\beta\rho^2 g^2}{\mu}\left(\left(\frac{\partial \phi^*}{\partial x}\right)^2 + \left(\frac{\partial \phi^*}{\partial y}\right)^2 + \left(\frac{\partial \phi^*}{\partial z}\right)^2\right)$$

$$- \frac{k\beta\rho^2 g^2}{\mu}\frac{\partial \phi^*}{\partial z} + \frac{k\beta\rho^2 g^2}{\mu} + \frac{k\beta\rho^2 g^2}{\mu}\left(\left(\frac{\partial \phi^*}{\partial x}\right)^2 + \left(\frac{\partial \phi^*}{\partial y}\right)^2 + \left(\frac{\partial \phi^*}{\partial z} - 1\right)^2\right)$$

$$+ \frac{2kg\rho\beta}{\mu}\rho g\left(\frac{\partial \phi^*}{\partial z} - 1\right) = n\beta\rho g\frac{\partial \phi^*}{\partial t} \quad \rightarrow (3)$$

Cancel or eliminate ρg throughout equation (3) we get:

$$\frac{k}{\mu}\left(\frac{\partial^2 \phi^*}{\partial x^2} + \frac{\partial^2 \phi^*}{\partial y^2} + \frac{\partial^2 \phi^*}{\partial z^2}\right) + \frac{k\beta\rho g}{\mu}\left(\left(\frac{\partial \phi^*}{\partial x}\right)^2 + \left(\frac{\partial \phi^*}{\partial y}\right)^2 + \left(\frac{\partial \phi^*}{\partial z}\right)^2\right]$$

$$- \frac{k\beta\rho g}{\mu}\frac{\partial \phi^*}{\partial z} + \cancel{\frac{k\beta\rho g}{\mu}} + \frac{k\beta\rho g}{\mu}\left[\left(\frac{\partial \phi^*}{\partial x}\right)^2 + \left(\frac{\partial \phi^*}{\partial y}\right)^2 + \left(\frac{\partial \phi^*}{\partial z}\right)^2\right] - \frac{2k\beta\rho g}{\mu}\frac{\partial \phi^*}{\partial z}$$

$$+ \cancel{\frac{k\beta\rho g}{\mu}} + \frac{2k\beta\rho g}{\mu}\frac{\partial \phi^*}{\partial z} - \cancel{\frac{2k\beta\rho g}{\mu}} = n\beta\frac{\partial \phi^*}{\partial t}$$

i.e.

$$\frac{k}{\mu}\nabla^2\phi^* + \frac{2k\beta\rho g}{\mu}\left[(\nabla\phi^*)^2\right] - \frac{k\beta\rho g}{\mu}\frac{\partial\phi^*}{\partial z} = n\beta\frac{\partial\phi^*}{\partial t} \quad \hookrightarrow (4)$$

If we define hydraulic conductivity $K = \frac{k\rho g}{\mu}$ = a constant

Then

$$\boxed{K\,\nabla^2\phi^* + 2K\beta\rho g\,(\nabla\phi^*)^2 - K\beta\rho g\,\frac{\partial\phi^*}{\partial z} = n\beta\rho g\,\frac{\partial\phi^*}{\partial t}} \quad \hookrightarrow (5)$$

If we neglect higher degree differential terms, then equation (5) reduces to:

$$K\,\nabla^2\phi^* - K\beta\rho g\,\frac{\partial\phi^*}{\partial z} = n\beta\rho g\,\frac{\partial\phi^*}{\partial t}$$

Mass Conservation of Homogeneous Compressible Fluid in a Deforming Porous Media

Earlier we were discussing about continuity equation in a non-deforming porous media.

Recall the general continuity equation for a homogeneous fluid:

$$\nabla\cdot(\rho\vec{q}) + \frac{\partial}{\partial t}(n\rho) = 0$$

i.e.

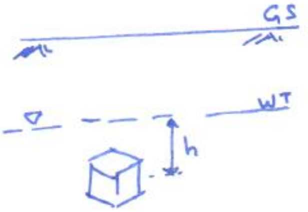
$$\frac{\partial}{\partial x}(\rho q_x) + \frac{\partial}{\partial y}(\rho q_y) + \frac{\partial}{\partial z}(\rho q_z) + n\frac{\partial \rho}{\partial t} + \rho\frac{\partial n}{\partial t} = 0$$

Now here: $\rho \frac{\partial n}{\partial t} \neq 0$ because of consolidation or deformation of porous media.

⇒ In confined aquifers when the water is pumped, there are chances of consolidation

Matrix & Medium Compressibilities

A porous medium volume at some depth below ground surface is subjected to – hydrostatic pressure (p) and external stress (σ) exerted by the porous formation above it.



Consolidation of the porous medium can occur.

Coefficient of bulk compressibility – α_b' w.r.t. external stress σ

→ In a saturated porous medium, fractional change in bulk volume of the porous medium with unit change in σ.

$$\alpha_b' = -\frac{1}{U_b}\frac{dU_b}{d\sigma}\bigg|_{p = \text{constant}}$$

U_b → volume of fixed mass of the porous medium.

Most of the time σ remains approximately same throughout.

We can define coefficient of bulk compressibility (α_b) with respect to liquid pressure (p).

$$\alpha_b = -\frac{1}{U_b}\frac{dU_b}{dp}\bigg|_{\sigma = \text{constant}}$$

As the volume of porous medium $U_b = U_s + U_p$

$U_s \rightarrow$ volume of rock matrix

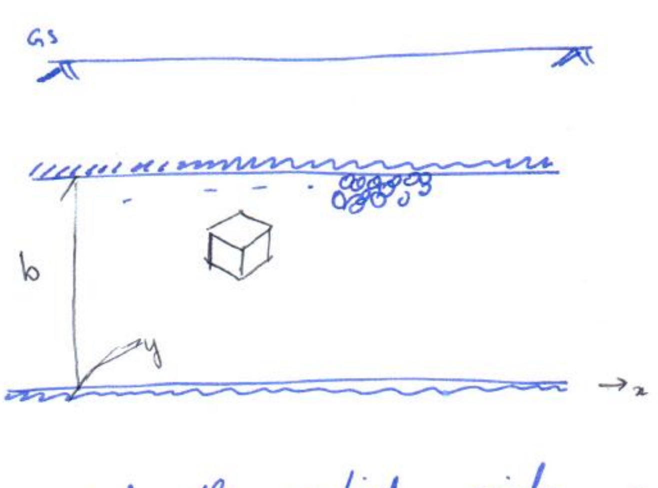
$U_p \rightarrow$ volume of pores

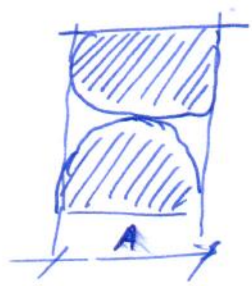
We can also define rock compressibility α_s

$$\alpha_s = -\frac{1}{U_s}\frac{dU_s}{dp}\bigg|_{\sigma = \text{constant}}$$

or Pore compressibility, $\alpha_p = -\frac{1}{U_p}\frac{dU_p}{dp}\bigg|_{\sigma = \text{constant}}$

Now consider a confined aquifer as shown in fig. below.





→ In the confined aquifer - let us neglect molecular and inter-particular forces among individual grains.

→ Moreover we assume individual grains are not compressible.

→ From the expanded cross-sectional area A

Solid-solid contact area $= mA$

Water solid contact area $= (1-m)A$

→ If p is pressure in water

σ_s → stress in solid

Then total ~~stress~~ force balance, $\sigma A = p(1-m)A + \sigma_s mA$

or Total stress, $\sigma = p(1-m) + \sigma_s m$

Mostly $m << 1.0$

The quantity $m\sigma_s$ = defined as effective stress σ'

$$\therefore \sigma = \sigma' + (1-m)p$$

Also $(1-m)p \approx p$

$$\therefore \boxed{\sigma = \sigma' + p}$$

Effective stress concept.

Now changes in porosity will be due to

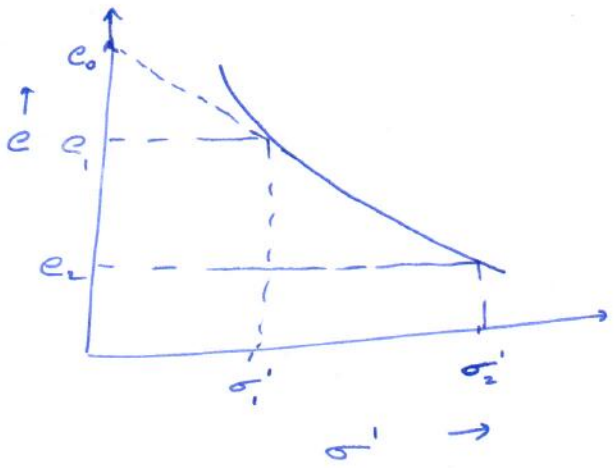
→ Variations in effective stress, σ'

→ Variations in water pressure, p

$$d\sigma = d\sigma' + dp$$

Void ratio $e = \frac{n}{1-n}$

For one-dimensional analysis, the relation between void ratio and effective stress can be plotted:



$e = e_0 - \alpha^* \sigma'$

where $\alpha^* \rightarrow$ slope of the straight line in the graph.

$\alpha^* \rightarrow$ Compressibility coefficient of soil

$$e = e_0 - \alpha^* \sigma'$$

$$\therefore \frac{\partial e}{\partial t} = -\alpha^* \frac{\partial \sigma'}{\partial t}$$

i.e. $$\frac{\partial}{\partial t}\left(\frac{n}{1-n}\right) = \frac{1}{1-n}\frac{\partial n}{\partial t} + \frac{n}{(1-n)^2}\frac{\partial n}{\partial t}$$

$$= \frac{1}{(1-n)^2}\frac{\partial n}{\partial t}$$

i.e. $$\frac{1}{(1-n)^2}\frac{\partial n}{\partial t} = -\alpha^* \frac{\partial \sigma'}{\partial t}$$

$$= -\alpha^* \left(\frac{\partial \sigma}{\partial t} - \frac{\partial p}{\partial t}\right)$$