CONTINUITY EQUATION FOR HOMOGENEOUS FLUIDS IN POROUS MEDIA

In the last class, we derived the continuity equation for a homogeneous fluid compressible fluid in an anisotropic porous media using control volume approach:

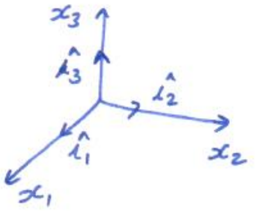
i.e in index notation:

$$\frac{\partial}{\partial x_i}(\rho q_i) + \frac{\partial}{\partial t}(n\rho) = 0 \quad \rightarrow (1)$$

Subsequently for a non-deforming porous media eqn. (1) modifies to

$$\frac{\partial}{\partial x_i}(\rho q_i) + n\frac{\partial \rho}{\partial t} = 0 \quad \rightarrow (2)$$

Following the same co-ordinate system:



Assuming the coordinates are so chosen that they are in principal directions for the permeability tensor.

$$\text{Then } \quad k_{ij} = \begin{pmatrix} k_{11} & 0 & 0 \\ 0 & k_{22} & 0 \\ 0 & 0 & k_{33} \end{pmatrix}$$

The density of the fluid (homogeneous) will be function of only pressure $\rho(p)$.

Recall, Darcy velocity $q_i = \frac{-k_{ij}\,\rho g}{\mu}\frac{\partial \phi^*}{\partial x_j}$

where, $\phi^* \rightarrow$ piezometric head for compressible fluids

See that,

$\phi = \frac{p}{\rho g} + z$, only for incompressible fluids

$$\cancel{\therefore q_i = \frac{-k_{ij}}{\mu}\left(\frac{\partial}{\partial x_j}\left(\frac{p}{\rho g}\right) + \frac{\partial}{\partial x_j}(z)\right)\rho g}$$

Substituting this relation in equation (2):

i.e. $$\frac{\partial}{\partial x_i}\left(\frac{\rho^2 g\, k_{ij}}{\mu}\frac{\partial}{\partial x_j}\phi^*\right) = n\frac{\partial \rho}{\partial t} \rightarrow (3)$$

where $k_{ij} = 0$ for $i \neq j$

* <u>To describe piezometric head for compressible fluids</u>

For compressible fluids, $\rho = \rho(p)$

The potential or piezometric head is given as

$$g\,\phi^* = gz + \int_{p_0}^{p}\frac{dp}{\rho}$$

$$g\,\frac{\partial \phi^*}{\partial x_i} = g\,\frac{\partial z}{\partial x_i} + \frac{\partial}{\partial x_i}\left(\int_{p_0}^{p}\frac{dp}{\rho}\right)$$

For irrotational flows, we have

$$g \frac{\partial \phi^*}{\partial x_i} = g \frac{\partial z}{\partial x_i} + \frac{1}{\rho} \frac{\partial p}{\partial x_i}$$

∴ In equation 3, we have then:

$$\frac{\partial}{\partial x_i}\left(\frac{\rho^2 g}{\mu} k_{ij} \left(\frac{\partial z}{\partial x_j} + \frac{1}{\rho g} \frac{\partial p}{\partial x_j} \right)\right) = n \frac{\partial \rho}{\partial t} \rightarrow (4)$$

In expanded form:

$$\frac{\partial}{\partial x_1}\left(\frac{\rho}{\mu} k_{11} \frac{\partial p}{\partial x_1} \right) + \frac{\partial}{\partial x_2}\left(\frac{\rho}{\mu} k_{22} \frac{\partial p}{\partial x_2} \right) + \frac{\partial}{\partial x_3}\left(\frac{k_{33} \rho}{\mu} \left(\frac{\partial z}{\partial x_3} \rho g + \frac{\partial p}{\partial x_3} \right)\right)$$
$$= n \frac{\partial \rho}{\partial t}$$

or

in x, y, z co-ordinate system:

$$\frac{\partial}{\partial x}\left(\frac{\rho}{\mu} k_x \frac{\partial p}{\partial x} \right) + \frac{\partial}{\partial y}\left(\frac{\rho}{\mu} k_y \frac{\partial p}{\partial y} \right) + \frac{\partial}{\partial z}\left(\frac{k_z \rho}{\mu} \left[\rho g + \frac{\partial p}{\partial z} \right]\right) = n \frac{\partial \rho}{\partial t}$$

For homogeneous and isotropic porous medium:

$$\frac{\partial}{\partial x_i}(k_{ij}) = 0 \quad , \quad k_{ij} = k \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

We have:

$$k \left[\frac{\partial}{\partial x}\left(\frac{\rho}{\mu} \frac{\partial p}{\partial x} \right) + \frac{\partial}{\partial y}\left(\frac{\rho}{\mu} \frac{\partial p}{\partial y} \right) + \frac{\partial}{\partial z}\left(\frac{\rho}{\mu} \left(\frac{\partial p}{\partial z} + \rho g \right)\right)\right] = n \frac{\partial \rho}{\partial t}$$

Assuming $\quad K = \frac{k\rho g}{\mu} \approx$ a constant

We have:

$K \cancel{\left[\frac{1}{g}\frac{\partial}{\partial x}(\right.}$

$$K\left[\frac{1}{g}\frac{\partial^2 p}{\partial x^2} + \frac{1}{g}\frac{\partial^2 p}{\partial y^2} + \frac{1}{g}\frac{\partial^2 p}{\partial z^2} + \frac{\partial \rho}{\partial z}\right] = n\frac{\partial \rho}{\partial t}$$

But we have $\rho = \rho(p)$

$$K\left[\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2} + g\frac{\partial \rho}{\partial z}\right] = ng\frac{\partial \rho}{\partial t} \rightarrow (5)$$

You have to define everytime $\rho = \rho(p)$.

Continuity Equation for Incompressible Fluid

i.e. ρ = a constant

In a non-deforming porous media

$$\nabla \cdot \vec{q} = 0$$

or $$\frac{\partial}{\partial x_i} q_i = 0$$

For ρ = constant, μ = constant

the fluid is incompressible and homogeneous.

$\therefore$ Piezometric head, $\quad \phi = \frac{p}{\rho g} + z$

$$\underline{\underline{K}} = \frac{\underline{\underline{k}}\,\rho g}{\mu}$$

If the x, y, z directions are principal directions

Then

$$\frac{\partial}{\partial x_i}\left(-K_{ij}\frac{\partial \phi}{\partial x_j}\right) = 0$$

i.e. $$\frac{\partial}{\partial x}\left(K_x \frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(K_y \frac{\partial \phi}{\partial y}\right) + \frac{\partial}{\partial z}\left(K_z \frac{\partial \phi}{\partial z}\right) = 0 \quad \rightarrow (6)$$

If the porous medium is homogeneous and isotropic:

That means $K_x = K_y = K_z = K$

$$\frac{\partial K}{\partial x} = 0 = \frac{\partial K}{\partial y} = \frac{\partial K}{\partial z}$$

∴ Equation (6) becomes:

$$K\left[\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}\right] = 0$$

or $$\boxed{\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0} \quad \rightarrow (7)$$

The Laplace equation

So note that we can use Laplace equation to solve for piezometric head only for homogeneous incompressible fluid flow in homogeneous - isotropic porous medium.

Q: What happens in this situation for fluid flow
→ Whether it will be steady or unsteady?

Continuity Equation for a Compressible Fluid

Let β = fluid compressibility term

$$\beta = -\frac{1}{U}\frac{DU}{Dp}$$

where $U \rightarrow$ volume of fluid
$p \rightarrow$ pressure of fluid

Also $\beta = \frac{1}{\rho}\frac{D\rho}{Dp}$ for iso-thermal conditions

(Compressibility $\rightarrow$ Measure of volume (or density) changes when a fluid is subjected to tension and compressions (i.e. normal stresses)).

Usually density of fluid depends on pressure and temperature.
In isothermal conditions $\beta = \frac{1}{\rho}\frac{\partial \rho}{\partial p}$; $T =$ constant

If pressure is constant, then $\beta = -\frac{1}{\rho}\frac{\partial \rho}{\partial T}$; $p =$ constant

If β is independent of pressure:

Volume $U = U_0 \exp\left[-\beta(p - p_0)\right]$

where U_0, p_0 $\rightarrow$ values at reference.

We also get: $\rho = \rho_0 \exp\left[\beta(p - p_0)\right]$

for β independent of pressure.

$\rho_0 \rightarrow$ density at reference pressure p_0

We have seen for isothermal conditions:

$$\beta = \frac{1}{\rho}\frac{\partial \rho}{\partial p}$$

$$\therefore \frac{\partial \rho}{\partial p} = \rho\beta$$

We can express:

$$\frac{\partial \rho}{\partial x} = \rho\beta\frac{\partial p}{\partial x} \quad \text{and} \quad \frac{\partial \rho}{\partial t} = \rho\beta\frac{\partial p}{\partial t}$$

$$\left(\text{i.e} \quad \text{grad}\ \rho = \rho\beta\ \text{grad}\ p \quad \text{or} \quad \nabla\rho = \rho\beta\nabla p\right)$$

Recall the continuity equation for homogeneous compressible fluid in homogeneous isotropic porous media:

$$k\left[\frac{\partial}{\partial x}\left(\frac{\rho}{\mu}\frac{\partial p}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\rho}{\mu}\frac{\partial p}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\rho}{\mu}\left(\frac{\partial p}{\partial z} + \rho g\right)\right)\right] = n\frac{\partial \rho}{\partial t} \longrightarrow (c)$$

As the fluid is homogeneous, we assume $\mu = \text{constant}$

$$\therefore \frac{k\rho}{\mu}\left(\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2}\right) + \frac{k}{\mu}\left(\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial x} + \frac{\partial p}{\partial y}\frac{\partial \rho}{\partial y} + \frac{\partial \rho}{\partial z}\frac{\partial p}{\partial z}\right)$$

$$+ \frac{k\rho g}{\mu}\frac{\partial \rho}{\partial z} \times 2 = n\frac{\partial \rho}{\partial t} \longrightarrow (8)$$

$$\text{i.e.} \quad \frac{k\rho}{\mu}\nabla^2 p + \frac{k}{\mu}\rho\beta(\nabla p)^2 + \frac{2k\rho g}{\mu}\rho\beta\frac{\partial p}{\partial z} = n\rho\beta\frac{\partial p}{\partial t}$$

or

$$\frac{k}{\mu}\nabla^2 p + \frac{k\beta}{\mu}(\nabla p)^2 + \frac{2k\rho g\beta}{\mu}\frac{\partial p}{\partial z} = n\beta\frac{\partial p}{\partial t} \longrightarrow (9)$$

where $k \rightarrow$ homogeneous isotropic permeability
$\beta \rightarrow$ compressibility of fluid in iso-thermal conditions

Equation (9) is the continuity equation of a homogeneous compressible fluid in a homogeneous isotropic porous media.

The dependent variable in the expression is pressure p.

⇒ We can also write the continuity equation in terms of piezometeric head ϕ^*.

Continuity equation using piezometeric head ϕ^*

Recall we had stated earlier for compressible fluid

$$g\,\phi^* = gz + \int_{p_0}^{p} \frac{dp}{\rho(p)}$$

i.e. $$\phi^* = z + \int_{p_0}^{p} \frac{dp}{g\,\rho(p)}$$

Also $$\rho = \rho_0 \exp\left[\beta(p - p_0)\right]$$

i.e. $$\phi^* = z + \frac{1}{g\rho_0} \int_{p_0}^{p} \frac{dp}{e^{\beta p} \cdot e^{-\beta p_0}}$$

i.e. $$\phi^* = \frac{1}{g\rho_0\, e^{-\beta p_0}} \int_{p_0}^{p} e^{-\beta p}\, dp$$

$$\therefore \quad \phi^* = z + \frac{e^{\beta p_0}}{g\beta\rho_0}\left(-e^{-\beta p} + e^{-\beta p_0}\right)$$

or $$\phi^* = z + \left(\frac{1}{g\beta\rho_0} - \frac{e^{\beta p_0}e^{-\beta p}}{g\beta\rho_0}\right)$$

$$= z + \frac{1}{g\beta\rho_0} - \frac{1}{g\beta\rho_0 \exp[\beta(p-p_0)]}$$

$$\phi^* = z + \frac{1}{g\beta}\left(\frac{1}{\rho_0} - \frac{1}{\rho}\right) \rightarrow (10)$$

Now differentiating.

$$g\frac{\partial\phi^*}{\partial x} = -\frac{1}{\beta}\frac{\partial}{\partial x}\left(\frac{1}{\rho}\right) = \frac{1}{\beta\rho^2}\frac{\partial\rho}{\partial x} \equiv \frac{1}{\rho}\frac{\partial p}{\partial x}$$

lly $$g\frac{\partial\phi^*}{\partial y} = \frac{1}{\rho}\frac{\partial p}{\partial y}$$

and $$g\frac{\partial\phi^*}{\partial z} = \frac{\partial}{\partial z}\left(gz - \frac{1}{\beta\rho}\right)$$

$$\equiv g + \frac{1}{\rho}\frac{\partial p}{\partial z}$$

Assuming datum is not changing w.r.t time,

$$g\frac{\partial\phi^*}{\partial t} = -\frac{1}{\beta}\frac{\partial}{\partial t}\left(\frac{1}{\rho}\right) = \frac{1}{\beta\rho^2}\frac{\partial\rho}{\partial t} \equiv \frac{1}{\rho}\frac{\partial p}{\partial t}$$

$\therefore$ The continuity equation (c):

$$k\left[\frac{\partial}{\partial x}\frac{\rho}{\mu}\frac{\partial p}{\partial x} + \frac{\partial}{\partial y}\left(\frac{\rho}{\mu}\frac{\partial p}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\rho}{\mu}\left(\frac{\partial p}{\partial z} + \rho g\right)\right)\right] = n\frac{\partial\rho}{\partial t}$$

can be converted in terms of ϕ^*.