LECTURE - 18
17 - FEB - 2014

Anisotropic Hydraulic Conductivity

In the last class, we discussed on hydraulic conductivity in → isotropic porous
(of homogeneous fluid) → anisotropic porous media

For anisotropic porous media we can represent Darcy's equation :

$$\vec{q} = \underline{\underline{K}} \cdot \vec{J}$$

$$\text{or} \quad q_i = - K_{ij} \frac{\partial \phi}{\partial x_j}$$

The hydraulic conductivity is a second-rank tensor having components :

$$\underline{\underline{K}} = \begin{pmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{pmatrix}$$

Like any tensor like stress, strain, etc that describe physical phenomena, hydraulic conductivity (as well as permeability) ~~are~~ is a symmetric tensor.

We subsequently suggested that we can describe principal values and principal directions for hydraulic conductivity.

Q: What is the significance of principal values (and/or principal directions)?

⇒ ~~The~~ In the porous media, we can have directions through which the magnitude of hydraulic conductivity is maximum. These directions are therefore, critical in designing various subsurface hydrological features (e.g. Recharge of aquifers; Pollution control; etc.) These directions are irrespective of the coordinate system adopted.

⇒ For a given K_{ij};
The principal values are obtained by solving the third degree polynomial

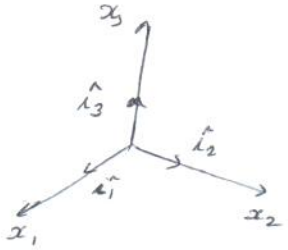
$$|K_{ij} - K\delta_{ij}| = 0 \quad \rightarrow (1)$$

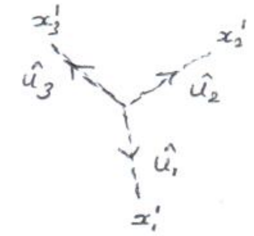
where K is a scalar value.

→ This characteristic polynomial on solving provides three values for K_{ij} - $K^{(1)}, K^{(2)}, K^{(3)}$.
These three values are three principal values of the given hydraulic conductivity tensor.

→ The directions to these principal values are given:
i.e. $K^{(1)} \rightarrow \hat{u}^{(1)}$; $K^{(2)} \rightarrow \hat{u}^{(2)}$; $K^{(3)} \rightarrow \hat{u}^{(3)}$

If $K^{(1)}, K^{(2)}, K^{(3)}$ are distinct values,
then $\hat{u}^{(1)}, \hat{u}^{(2)}$, and $\hat{u}^{(3)}$ are mutually orthogonal.





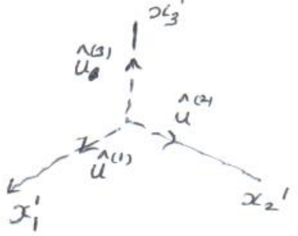
→ If we now represent the hydraulic conductivity in the new coordinate (x_1', x_2', x_3') system whose unit vectors are $\hat{u}^{(1)}, \hat{u}^{(2)}, \hat{u}^{(3)}$,

then the hydraulic conductivity

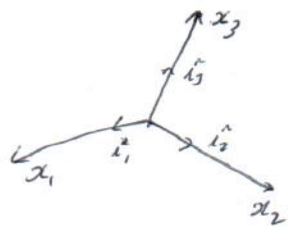
$$K_{ij}' = \begin{pmatrix} K^{(1)} & 0 & 0 \\ 0 & K^{(2)} & 0 \\ 0 & 0 & K^{(3)} \end{pmatrix}$$

⇒ We can also ~~th~~ extract information from reverse process
If you have hydraulic conductivity represented in principal directions and values, then we can have hydraulic conductivity tensor in any arbitrary coordinate system as follows:

Given $K_{ij}' = \begin{pmatrix} K^{(1)} & 0 & 0 \\ 0 & K^{(2)} & 0 \\ 0 & 0 & K^{(3)} \end{pmatrix}$



In the coordinate system



$$K_{ij} = K^{(1)} u_i^{(1)} u_j^{(1)} + K^{(2)} u_i^{(2)} u_j^{(2)} + K^{(3)} u_i^{(3)} u_j^{(3)}$$

i.e. You have components

$$K_{11} = K^{(1)} u_1^{(1)} u_1^{(1)} + K^{(2)} u_1^{(2)} u_1^{(2)} + K^{(3)} u_1^{(3)} u_1^{(3)}$$

$$K_{12} = K^{(1)} u_1^{(1)} u_2^{(1)} + K^{(2)} u_1^{(2)} u_2^{(2)} + K^{(3)} u_1^{(3)} u_2^{(3)} \text{ ; etc.}$$

$\Rightarrow$ You can also use Mohr's circle for finding principal directions and values.

$\Rightarrow$ This way you can represent hydraulic conductivity using principal values.

$$\underline{\underline{K}} = \begin{pmatrix} K^{(1)} & 0 & 0 \\ 0 & K^{(2)} & 0 \\ 0 & 0 & K^{(3)} \end{pmatrix}$$

Then Darcy discharge will be:

$$\vec{q} = \underline{\underline{K}} \cdot \vec{J}$$

Here
$$q_1 = -K^{(1)} \frac{\partial \phi}{\partial x_1}$$

$$q_2 = -K^{(2)} \frac{\partial \phi}{\partial x_2}$$

$$q_3 = -K^{(3)} \frac{\partial \phi}{\partial x_3}$$

Directional Permeability or Hydraulic Conductivity

The darcy discharge $\vec{q} = \underline{\underline{K}} \cdot \vec{J}$

In the expression there are two vectors $\vec{q}$ and $\vec{J}$ (i.e. Specific discharge vector and the negative of hydraulic gradient vector $\vec{J}$)

In general for an anisotropic porous media

$$\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} = - \begin{pmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{pmatrix} \begin{Bmatrix} \partial\phi/\partial x_1 \\ \partial\phi/\partial x_2 \\ \partial\phi/\partial x_3 \end{Bmatrix}$$

i.e. $q_i = - K_{ij} \dfrac{\partial \phi}{\partial x_j}$

The vectors $\vec{q}$ and $\vec{J}$ are non-colinear for anisotropic porous medium.

→ They can be colinear, if $\vec{J}$ is in the direction of principal axes.

$$\therefore \quad \vec{q} \cdot \vec{J} = |\vec{q}||\vec{J}| \cos\theta$$

$$\text{or} \quad \cos\theta = \frac{\vec{q} \cdot \vec{J}}{|\vec{q}||\vec{J}|}$$

(i) Directional hydraulic conductivity in the direction of flow:

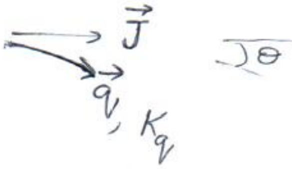
→ It is hydraulic conductivity in the direction of flow.

i.e. For an anisotropic porous medium :

$$q_i = K_{ij} \, J_j$$

The directional hydraulic conductivity is the ratio between specific discharge and component of hydraulic gradient in the direction of $\vec{q}$.

i.e. $$|\vec{q}| = K_q \, |\vec{J}| \cos\theta$$



or $$K_q = \frac{|\vec{q}|}{|\vec{J}| \cos\theta}$$

(ii) Directional hydraulic conductivity in the direction of hydraulic gradient.

→ Ratio between specific discharge $|\vec{q}|$ in the direction of hydraulic gradient and the gradient itself.

i.e. $$|\vec{q}| \cos\theta = K_J \, |\vec{J}|$$

or $$K_J = \frac{|\vec{q}| \cos\theta}{|\vec{J}|}$$

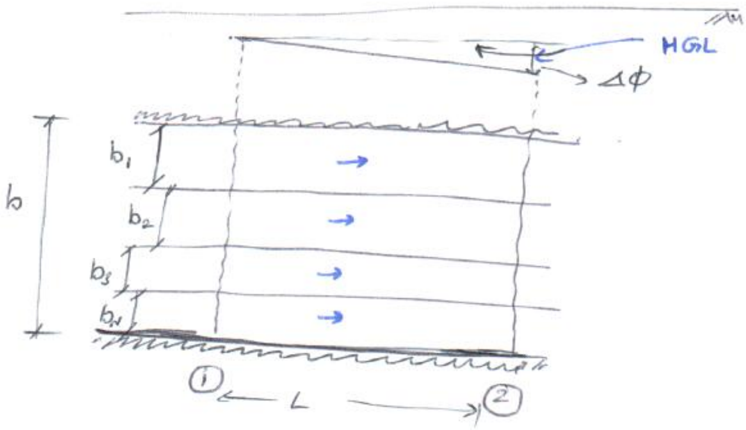
Hydraulic Conductivity in Layered Porous Media

In nature, the subsurface may exist as different layers. That is, the aquifers may not be homogeneous as well as they be anisotropic.

The subsurface zones may be highly complicated to evaluate hydraulic conductivity in each layers. In such situations, to evaluate Darcy's discharge, we can use equivalent hydraulic conductivities.

* The liquid considered is homogeneous, therefore, we can evaluate equivalent hydraulic conductivity (and not permeability).

(i) Flow parallel to combination of N- layers :



HGL → hydraulic gradient line

Let us consider a confined aquifer of thickness 'b' actually comprised of N - layers as shown in the figure.

Recall the discharge

$$Q = - \underline{\underline{K}} A \cdot \nabla \phi$$

∴ Flow in each layer per unit width of the aquifer can be given as product of hydraulic conductivity

$$\cancel{Q_i = K_i b}$$

the depth, and hydraulic gradient.

Let the ~~hydraulic~~ individual layer $I = 1, 2, 3, \dots, N$ be isotropic. The hydraulic conductivity of layer I is K_I.

We have now

$$Q_I = K_I \, b_I \, \frac{\Delta\phi}{L}$$

Total discharge from aquifer per unit horizontal width perpendicular to this paper as shown in earlier figure

$$Q = \sum_{I=1}^{N} Q_I \quad ; \quad b = \sum_{I=1}^{N} b_I$$

The hydraulic gradient $J = \dfrac{\phi_1 - \phi_2}{L} = -\dfrac{(\phi_2 - \phi_1)}{L}$

$$Q = \sum_{I=1}^{N} K_I \, b_I \, \frac{\Delta\phi}{L}$$

This hydraulic gradient $\frac{\Delta\phi}{L}$ is applicable throughout the domain.

$$\therefore \quad Q = \frac{\Delta\phi}{L} \sum_{I=1}^{N} K_I \, b_I$$

We can define $K_I \, b_I = T_I$ Transmissivity of I^{th} layer.

We can also define equivalent hydraulic conductivity $\bar{K}^P$ that give same flow Q for aquifer of depth b.

i.e. $Q = \bar{K}^P b \dfrac{\Delta\phi}{L} = \dfrac{\Delta\phi}{L} \displaystyle\sum_{I=1}^{N} K_I b_I$

or $$\boxed{\bar{K}^P = \frac{\sum_{I=1}^{N} K_I \, b_I}{\sum_{I=1}^{N} b_I}} \rightarrow (2)$$