LECTURE 17
14-FEB-2014

DARCY's LAW

We have seen in last class, the equation of motion of water (or fluid) in porous media can be expressed using Darcy's law.

Several forms or notations were used

e.g.: 1) $\vec{q} = K J$ (for homogeneous fluid)

2) For homogenous fluid in isotropic porous media

$$\vec{q} = -K \nabla \phi$$

or

$$q_i = -K \frac{\partial \phi}{\partial x_i}$$

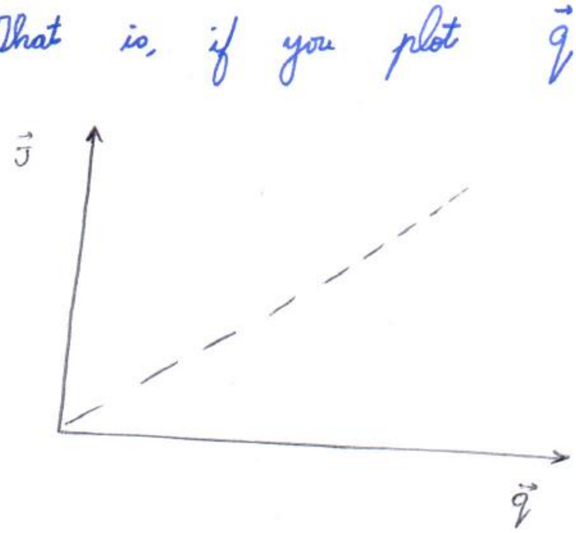
3) For homogeneous fluid in anisotropic porous media

$$q_i = -K_{ij} \frac{\partial \phi}{\partial x_j} \quad \text{or} \quad \vec{q} = -\underline{\underline{K}} \cdot \nabla \phi$$

The Limits of Darcy's Law

In the above relation of specific discharge w.r.t the hydraulic gradient, we can see that in Darcy's experiment and subsequent column flow analyses, the specific discharge is linearly related to hydraulic gradient.

That is, if you plot $\vec{q}$ vs $\vec{J}$, it should be a straight line.



→ However there is a limit in validity of Darcy's law.

→ Recall about Reynold's number you have studied in fluid mechanics for flow through pipes. Flows were classified as laminar and turbulent. The critical Reynold's number for pipe flows $\left(Re = \frac{V \rho D}{\mu}\right)$ is approximately 2100.

⇒ We can use a similar concept of Reynold's number for flow of fluid through porous media.

$$Re = \frac{q \rho d}{\mu}$$

where d → some length of importance in porous matrix (maybe characteristic length)

We may consider the mean grain diameters as characteristic length (say d_{10} or d_{50}, etc.)

From various experimental ~~observations~~, ~~of~~ it was inferred that Darcy's law is valid upto

$R_e = 1$ to 10.

For $R_e < 1.0$ the Darcy's law is absolutely applicable

For $1.0 \le Re \le 10.0$, the Darcy's law is applicable with mild wobbly situations.

For $Re > 10.0$, Darcy's law is <u>not applicable</u>.

Darcy's law is applicable only for laminar flows.

⇒ There can also be <u>lower limit</u> for the validity of Darcy's law.

Say if Negative of hydraulic gradient, $J \le J_0$

where $J_0 \to$ a threshold value,

Then $q = 0$.

⇒ Some information related to specific discharge.

★ In fluid mechanics, you might have studied rotational flows and irrotational flows that are defined based on velocity vector and hydraulic gradient vector. (i.e. $\nabla \times \vec{V} = 0$ for irrotational flow; Vorticity vector, $\vec{\omega} = \frac{1}{2}(\nabla \times \vec{V})$, etc.)

For porous media flows one can suggest of irrotational flows as :

$$\nabla \times \vec{q} = 0$$

Vortex flows given by vorticity vector,

$$\vec{\omega} = \frac{1}{2}(\nabla \times \vec{q})$$

Hydraulic Conductivity - Isotropic Media

Darcy's law

$$\vec{q} = K J$$

where $K \rightarrow$ coefficient of proportionality

This coefficient is usually called hydraulic conductivity (It is the ease with which a fluid is transported through a porous media).

* For an isotropic porous medium, it is the specific discharge per unit hydraulic gradient.

→ This coefficient K depends on both fluid properties as well as matrix properties of porous media.

Recall $$K = \frac{k \rho g}{\mu}$$

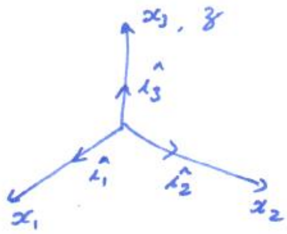
where $k \rightarrow$ permeability

For isotropic porous media,

$$q_i = -K \frac{\partial \phi}{\partial x_i}$$

$$= -\left(\frac{k \rho g}{\mu}\right) \frac{\partial}{\partial x_i}\left(\frac{p}{\rho g} + z\right)$$

Note: In the co-ordinate system



$$\vec{x} = x_1 \hat{i}_1 + x_2 \hat{i}_2 + x_3 \hat{i}_3$$

$$\therefore \frac{\partial x_3}{\partial x_i} \left(= \frac{\partial z}{\partial x_i} \right)$$

$$\equiv \left\{ \frac{\partial x_3}{\partial x_1} = 0 \; ; \quad \frac{\partial x_3}{\partial x_2} = 0 \; ; \quad \frac{\partial x_3}{\partial x_3} = 1 \right\}$$

Also let us assume acceleration due to gravity

$$\vec{g} = g_1 \hat{i}_1 + g_2 \hat{i}_2 + g_3 \hat{i}_3$$

If $\hat{i}_1$, $\hat{i}_2$ are horizontal axis, then $g_1 = g_2 = 0$

$$g_3 = -g$$

$$\therefore \vec{g} = -g \hat{i}_3$$

$$\therefore q_i = -\frac{k}{\mu} \frac{\partial}{\partial x_i}(p + \rho g z) \quad ; \quad i = 1, 2, 3$$

For horizontal flows, $\frac{\partial x_3}{\partial x_1} = \frac{\partial x_3}{\partial x_2} = 0$

$$\therefore q_i = -\frac{k}{\mu} \frac{\partial p}{\partial x_i}$$

$\Rightarrow$ The intrinsic permeability k depends on solid grain diameter and shape.

If k varies w.r.t space, that is $k \to k(x_1, x_2, x_3)$, then the porous media is heterogeneous.

As $K = \frac{k \rho g}{\mu}$.

For homogeneous fluid flow in porous media, it is mandatory to have the porous media also as homogeneous.

Units of K

For aquifers $\to$ m/d
m/yr

In soil engineering or irrigation, ~~etc~~ and drainage we may use $\to$ cm/s

Units of k

k has dimensions of L^2.

Common unit is $\to$ darcy

$$1 \text{ darcy} = 9.8697 \times 10^{-9} \text{ cm}^2$$

Usually for water at 20° C,

For $K = 1$ cm/s,

$$k = 1.02 \times 10^{-5} \text{ cm}^2$$

Anisotropic Permeability

Recall the Darcy's law expressed in index notation

$$q_i = K_{ij} J_j$$

where K_{ij} → the second rank hydraulic conductivity tensor

J_j → The negative of hydraulic gradient vector.

$$\vec{q} = \underline{\underline{K}} \cdot \vec{J}$$

or
$$q_i = -\left(\frac{k_{ij}\,\rho g}{\mu}\right) \frac{\partial}{\partial x_i}\left(\frac{p}{\rho g} + z\right)$$

Here k_{ij} → The second rank permeability tensor

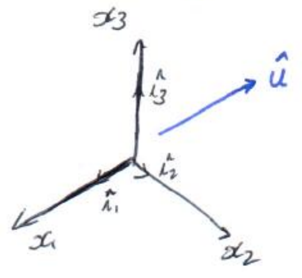
⇒ What ever properties you might have learned about a second-ranked tensor holds true for K_{ij} and k_{ij} also

(i) The tensor is independent of the co-ordinate system chosen.

i.e. From one co-ordinate we can get the components of the same tensor in another coordinate system by rotation.

(ii) If $\det(K_{ij})$ or $\det(k_{ij})$ are non-zero quantities, the reciprocal or conjugate tensor of $\underline{\underline{K}}$ & $\underline{\underline{k}}$ also exist.

★ We can also have principal components of hydraulic conductivity as follows:



→ We ~~have~~ have the three directions and $\underline{\underline{K}}$ will have nine components

↳ Consider a unit vector $\hat{u}$ as shown in the figure.

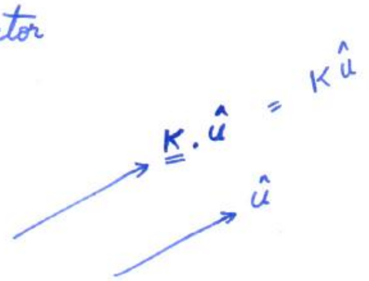
i.e. $\hat{u} = u_1 \hat{i}_1 + u_2 \hat{i}_2 + u_3 \hat{i}_3$

⇒ The direction $\hat{u}$ will become principal direction of $\underline{\underline{K}}$, if the product $\underline{\underline{K}} \cdot \hat{u}$ (i.e $K_{ij} u_j$) is parallel to $\hat{u}$.

Note: The dot product of second rank tensor and vector will give you a vector.

∴ $K_{ij} u_j$ is a vector

If this product's direction is parallel to $\hat{u}$,



Then the product $K_{ij} u_j$ can be given as a product of some scalar quantity K and the unit vector $\hat{u}$.

i.e. $K_{ij} u_j = K u_j$

i.e. $(K_{ij} - K \delta_{ij}) u_j = 0 \longrightarrow$ ①

The system $\left(K_{ij} - K\delta_{ij}\right) u_j = 0$

becomes a homogeneous system (refer numerical methods).

(Not about the soil or liquid properties)

It is an Eigen-value problem.

The determinant of that system

$$\left| K_{ij} - K\delta_{ij} \right| = 0$$

On expanding this ~~equation~~ determinant you will get a characteristic equation in K.

As we have choosen 3-dimensional coordinates, the characteristic equation will be a third degree polynomial in K

i.e
$$\begin{vmatrix} (K_{11} - K) & K_{12} & K_{13} \\ K_{21} & (K_{22} - K) & K_{23} \\ K_{31} & K_{32} & (K_{33} - K) \end{vmatrix} = 0$$

On solving this equation, we will get three Eigen values say $K^{(1)}, K^{(2)}, K^{(3)}$.

$\Rightarrow$ These three eigen values are principal values of hydraulic conductivity.

$\Rightarrow$ We can also now associate Eigen vectors for these Eigen values. Let the unit vectors

for these three Eigen values be $\hat{u}^{(1)}$, $\hat{u}^{(2)}$
and $\hat{u}^{(3)}$.

$\Rightarrow$ These three unit vectors will be mutually perpendicular
for distinct values of $K^{(1)}$, $K^{(2)}$, and $K^{(3)}$.