LECTURE 16

13 - FEB - 2014

EQUATION OF MOTION OF HOMOGENEOUS FLUID

Recall in the last class we discussed on conservation of mass and linear momentum of fluid in porous media.

→ Due to continuum assumptions for the porous media, the above conservation equations yielded parameters (or coefficients) like

* Molecular diffusion coefficient
* Mechanical dispersion coefficient
* Tortuosity
* Permeability
* Hydraulic conductivity, etc.

In the equation for conservation of linear momentum, we saw that:

$$q_i + \left(\frac{\bar{B}\,\bar{\rho}}{\bar{\mu}}\right)\frac{\partial q_i}{\partial t} = -\frac{k_{ij}}{\bar{\mu}}\left(\frac{\partial \bar{p}}{\partial x_j} + \bar{\rho} g \frac{\partial z}{\partial x_j}\right) \quad (1)$$

where

q_i → Specific discharge vector

$\bar{B}$ → Average conductance value over REV

$\bar{\rho}$ → Density averaged over REV

$\bar{\mu}$ → Viscosity averaged over REV

$\bar{p}$ → Average pressure of liquid in REV

k_{ij} → permeability tensor

Subsequently we suggested that as laminar flow is assumed in pore channels, the effect of ~~viscosity~~ viscous forces may be more compared to inertial forces. Therefore, the local acceleration term in equation (1) can be neglected. It becomes:

$$\cancel{\partial q_i} \quad q_i = -\frac{k_{ij}}{\bar{\mu}} \left(\frac{\partial \bar{p}}{\partial x_j} + \bar{\rho} g \frac{\partial z}{\partial x_j} \right) \longrightarrow (2)$$

For <u>homogeneous incompressible fluid</u>, equation (2) becomes:

$$q_i = -\frac{k_{ij} \bar{\rho} g}{\bar{\mu}} \frac{\partial}{\partial x_j} \left(\frac{\bar{p}}{\bar{\rho} g} + z \right)$$

or

$$q_i = -\frac{k_{ij} \bar{\rho} g}{\bar{\mu}} \frac{\partial \phi}{\partial x_j} \longrightarrow (3)$$

where $\phi \rightarrow$ piezometric head $= \dfrac{\bar{p}}{\bar{\rho} g} + z$

Chapter

EQUATION OF MOTION OF HOMOGENEOUS FLUID

So we have seen that for homogeneous incompressible fluid,

Specific discharge $q_i = -\frac{k_{ij}\bar{\rho}g}{\bar{\mu}}\frac{\partial\phi}{\partial x_j}$

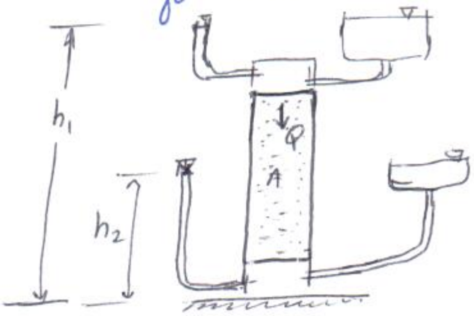
The quantity $\frac{k_{ij}\bar{\rho}g}{\bar{\mu}}$ is classified as hydraulic conductivity of the ~~poro~~ fluid in the porous media (K_{ij}).

$$\therefore \quad q_i = -K_{ij}\frac{\partial\phi}{\partial x_j}$$

$$\text{or} \quad \vec{q} = -\underline{\underline{K}}\cdot\nabla\phi \quad \rightarrow \quad (4)$$

→ This mathematical representation of specific discharge w.r.t. hydraulic conductivity was formulated only in 20th century.

→ However Henry Darcy (1856) had experimentally formulated the above relation much much earlier.



Darcy performed the flow of water through sand column.

He observed that the discharge (or rate of flow) of water from column is

- (i) proportional to the cross-sectional area A
- (ii) proportional to difference in hydraulic head (i.e. $h_1 - h_2$)
- (iii) inversely proportional to height or length of the column (L)

Using proportionality relations, Darcy's formula was:

$$Q = KA\frac{(h_1 - h_2)}{L}$$

where $K \rightarrow$ coefficient of proportionality

$h_1 \rightarrow$ piezometric head at top

$h_2 \rightarrow$ piezometric head at bottom

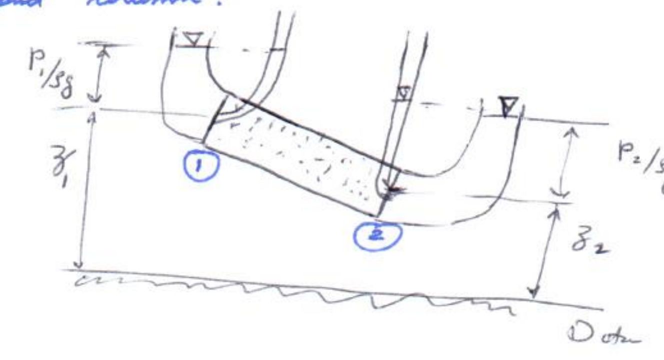
$\rightarrow$ The quantity $\frac{h_1 - h_2}{L}$ can be approximately interpreted as hydraulic gradient.

i.e. say $J = \frac{h_1 - h_2}{L}$

or $q = \frac{Q}{A} = KJ$

Consider the inclined sand column:

$\Rightarrow$ In this case what will be discharge?



$Q \propto A$

$Q \propto \Phi_1 - \Phi_2$

$Q \propto \frac{1}{L}$

$\Phi_1 \rightarrow$ piezometric head at 1

$\Phi_2 \rightarrow$ " " at 2

Note:

$$\Phi = \frac{p}{\rho g} + z$$

$$\therefore \; Q = KA \, \frac{\Phi_1 - \Phi_2}{L}$$

or $$q = K \, \frac{\Phi_1 - \Phi_2}{L}$$

Q: Do the fluid flow in porous media always occur from higher pressure to lower pressure?

$\Rightarrow$ We can now represent hydraulic gradient

$$J = \frac{\Phi_1 - \Phi_2}{L}$$

$$\therefore \; q = KJ$$

A

Recall porosity 'n'

We have seen that areal and volumetric porosities are same

In cross section A, the water flows through an area $= nA$

We can define subsequently average velocity or (seepage velocity)

$$V = \frac{Q}{nA} = \frac{q}{n}$$

$\Rightarrow$ If all the pores are not contributing to flow, due to presence of dead end pores, we may consider effective porosity n_e

Then we have seepage velocity $V = \dfrac{q}{n_e}$.

That way the Darcy's law suggest that flow of water in a porous media is proportional to area of cross section and hydraulic gradient.

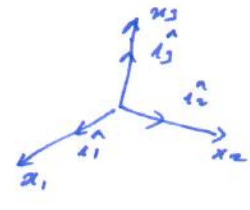
Generalisation of Darcy's Law

In an isotropic medium :

The three dimensional flow in porous media can be given as:

$$\vec{q} = K\vec{J} = -K\nabla\phi$$

(The negative sign is used because hydraulic ~~gradient~~ head decreases during flow).



$$\vec{q} = -K\nabla\phi$$

or $$q_i = -K\frac{\partial\phi}{\partial x_i}$$

Also $\vec{J} = -\nabla\phi$; $J_1 = -\dfrac{\partial\phi}{\partial x_1}$

$J_2 = -\dfrac{\partial\phi}{\partial x_2}$

$J_3 = -\dfrac{\partial\phi}{\partial x_3}$

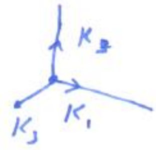
Do not confuse with diffusive flux defined in previous chapter

This is hydraulic gradient vector.

We can define specific discharge w.r.t its components in the fixed co-ordinate system

i.e. $\vec{q} = q_1 \hat{i_1} + q_2 \hat{i_2} + q_3 \hat{i_3}$

~~where $q_1 = -K\partial\phi$~~



For isotropic media,

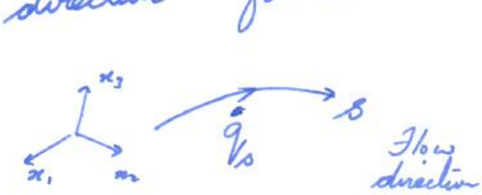
~~K~~, hydraulic conductivity is same in all three directions.

Let us call it $K_1 = K_2 = K_3 = K$

Then $q_1 = -K \dfrac{\partial \phi}{\partial x_1}$, $q_2 = -K \dfrac{\partial \phi}{\partial x_2}$

$$q_3 = -K \frac{\partial \phi}{\partial x_3}$$

$\Rightarrow$ We can also define the specific discharge along any arbitrary flow direction for such isotropic media:



$$q_s = -K \frac{\partial \phi}{\partial s}$$

$\Rightarrow$ Darcy's law can also be written as:

$$\vec{q} = -\frac{k}{\mu}\left[\frac{\partial p}{\partial x_1}\hat{i_1} + \frac{\partial p}{\partial x_2}\hat{i_2} + \left(\frac{\partial p}{\partial z} + \rho g\right)\hat{i_3}\right]$$

or $$q_i = -\frac{k}{\mu}\left(\frac{\partial p}{\partial x_j} + \rho g \frac{\partial z}{\partial x_j}\right)\delta_{ij}$$

$\delta_{ij} \rightarrow$ Kronecker Delta

$k \rightarrow$ permeability (same throughout for isotropic media).

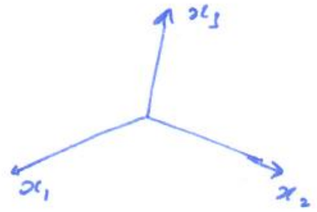
Anisotropic Media

For anisotropic porous medium, the permeabilities and hydraulic conductivities may differ in all directions.

For anisotropic media:

$$\vec{q} = K J \quad , \quad J = -\nabla\phi$$

$$= -K \nabla\phi$$



Now this is actually

$$q_1 = K_{11} J_1 + K_{12} J_2 + K_{13} J_3$$

$$q_2 = K_{21} J_1 + K_{22} J_2 + K_{23} J_3$$

$$q_3 = K_{31} J_1 + K_{32} J_2 + K_{33} J_3$$

$$\therefore \vec{q} = -\underline{\underline{K}} \cdot \nabla\phi$$

or $$q_i = -K_{ij} \frac{\partial\phi}{\partial x_j}$$