LECTURE 15
10 - FEB - 2014

CONSERVATION EQUATIONS IN POROUS MEDIA

Recalling the conservation equations described in the last classes (for porous media)

* The local volume conservation equation for an incompressible fluid is:

$$\left\langle \frac{\partial V_s'}{\partial s} \right\rangle = 0 \quad \rightarrow (1)$$

* The REV based volume conservation equation for an incompressible fluid in porous media:

$$\left.\begin{aligned} \frac{\partial}{\partial x_j}\left(n \bar{V}_j'\right) &= 0 \\ &\text{or} \\ \frac{\partial}{\partial x_j} q_j &= 0 \end{aligned}\right\} \rightarrow (2)$$

* Mass conservation equation for a species α in the solution (or fluid) in a porous medium

Assuming $I_\alpha = 0$

The local conservation equation is:

$$\left\langle \frac{\partial \rho_\alpha}{\partial t} \right\rangle = \left\langle \frac{\partial}{\partial \xi_i}\left(D_{\alpha\beta} T_{ij}^{*} \frac{\partial \rho_\alpha}{\partial \xi_j} \right) \right\rangle - \left\langle \frac{\partial}{\partial \xi_i}\left(\rho_\alpha V_i'\right) \right\rangle \quad \rightarrow (3)$$

* The mass conservation equation for species α in fluid in porous medium averaged over entire void space in REV..

$$\frac{\partial}{\partial t}\bar{\rho}_\alpha = \frac{\partial}{\partial x_i}\left(D_{\alpha\beta} \overline{T_{ij}^{*}} \frac{\partial \bar{\rho}_\alpha}{\partial x_j} \right) - \frac{\partial}{\partial x_i} \overline{\rho_\alpha V_i'} - \frac{\overline{\rho_\alpha V_i'}}{n} \frac{\partial n}{\partial x_i} \quad \rightarrow (4)$$

This REV averaged mass conservation equation for specie α can also be written in following form:

For incompressible liquid (assuming n = constant)

$$\frac{\partial}{\partial t}\bar{\rho}_\alpha = \frac{\partial}{\partial x_i}\left[\left(D_{ij} + D_{\alpha\beta}\overline{T^*_{ij}}\right)\frac{\partial}{\partial x_j}\bar{\rho}_\alpha\right] - \bar{V}'_i \frac{\partial \bar{\rho}_\alpha}{\partial x_i}$$

or $\hookrightarrow$ (5A)

Suggesting a second rank tensor say A_{ij} as $\underline{\underline{A}}$ symbolically.

$$\frac{\partial}{\partial t}\bar{\rho}_\alpha = \nabla\cdot\left[\left(\underline{\underline{D}} + D_{\alpha\beta}\underline{\underline{\overline{T^*}}}\right)\nabla\bar{\rho}_\alpha\right] - \vec{\bar{V}}'\cdot\nabla\bar{\rho}_\alpha \rightarrow (5B)$$

where $\underline{\underline{D}}$ → coefficient of mechanical dispersion tensor

$D_{\alpha\beta}\underline{\underline{\overline{T^*}}}$ → coefficient of molecular diffusion tensor

If $n \neq$ constant, then

$$\frac{\partial}{\partial t}\bar{\rho}_\alpha = \frac{\partial}{\partial x_i}\left[\left(D_{ij} + D_{\alpha\beta}\overline{T^*_{ij}}\right)\frac{\partial}{\partial x_j}\bar{\rho}_\alpha\right] - \bar{V}'_i\frac{\partial\bar{\rho}_\alpha}{\partial x_i} - \frac{\bar{\rho}_\alpha \overline{V'_i}}{n}\frac{\partial n}{\partial x_i}$$

* We can also have mass conservation for entire fluid

→ Averaged over cross sectional area:

$$\left\langle\frac{\partial\rho}{\partial t}\right\rangle = \left\langle\frac{\partial}{\partial\xi_i}\left[D_{\alpha\beta}T^*_{ij}\frac{\partial\rho}{\partial\xi_j}\right] - \frac{\partial}{\partial\xi_i}(\rho V'_i)\right\rangle$$

→ (6)

* The mass conservation of the entire multi-specie fluid averaged over the void spaces of the REV is given as:

(Assuming n = constant)

$$\frac{\partial \bar{\rho}}{\partial t} = \frac{\partial}{\partial x_i}\left[\left(D_{ij} + D_{\alpha\beta}\, \overline{T^*_{ij}}\right) \frac{\partial \bar{\rho}}{\partial x_j}\right] - \frac{\partial}{\partial x_i}\left(\bar{\rho}\, \bar{V}'_i\right) \quad \rightarrow (7)$$

* Similarly we can apply conservation of linear momentum also to the fluid in the void space in REV.

→ We can do local averaging

$$\left\langle V^*_i + \frac{B\rho}{\mu} \frac{\partial}{\partial t} V^*_i \right\rangle = -\left\langle \frac{B\, T^*_{ij}}{\mu} \left(\frac{\partial p}{\partial \xi_j} + \rho g \frac{\partial z}{\partial \xi_j} \right) \right\rangle \quad \rightarrow (8)$$

where

V^*_i → ~~Velocit~~ Mass averaged velocity vector

ρ → density of entire multi-species fluid

μ → dynamic viscosity

B → conductance of a channel. It's dimension L^2. (also function of the shape of channel's cross section).

* The REV averaged conservation of linear momentum is:

$$\overline{V^*_i} + \bar{B}\left(\overline{\frac{\rho}{\mu}}\right) \frac{\partial}{\partial t} \overline{V^*_i} = -\overline{B\, T^*_{ij}} \left[\frac{1}{\mu} \frac{\partial \bar{p}}{\partial x_j} + \frac{\bar{\rho}}{\mu} g \frac{\partial z}{\partial x_j} \right] \quad \rightarrow (9)$$

You may see that $n\, \overline{B\, T^*_{ij}} = k_{ij}$ permeability of the medium.

For liquids with its density fluctuation $\overset{\circ}{\rho}$ ~~such~~ such as $\left| \frac{\overset{\circ}{\rho}}{\bar{\rho}} \right| << 1$

Dynamic viscosity, $\left| \frac{\overset{\circ}{\mu}}{\bar{\mu}} \right| << 1$,

In such situations we can assume the momentum equation as:

$$\overline{V_i^*} + \frac{\bar{B}\,\bar{\rho}}{\bar{\mu}} \frac{\partial \overline{V_i^*}}{\partial t} = -\frac{k_{ij}}{n\bar{\mu}} \left(\frac{\partial \bar{p}}{\partial x_j} + \bar{\rho} g \frac{\partial z}{\partial x_j} \right) \rightarrow (10)$$

If the mass averaged velocity is replaced by volumetric averaged velocity $\vec{V}'$, then

$$\overline{V_i'} - D_{\alpha\beta} \overline{T_{ij}^*} \frac{1}{\rho} \frac{\partial \rho}{\partial x_j} + \frac{\bar{B}\,\bar{\rho}}{\bar{\mu}} \frac{\partial}{\partial t} \left[\overline{V_i'} - D_{\alpha\beta} \overline{T_{ij}^*} \frac{1}{\rho} \frac{\partial \rho}{\partial x_j} \right]$$

$$= -\frac{k_{ij}}{n\bar{\mu}} \left(\frac{\partial \bar{p}}{\partial x_j} + \bar{\rho} g \frac{\partial z}{\partial x_j} \right) \rightarrow (11)$$

If the flux due to molecular diffusion based on density gradient is such that:

$$\left| D_{\alpha\beta} T_{ij}^* \frac{\partial \bar{\rho}}{\partial x_j} \right| << \left| \bar{\rho}\, \bar{V}_i^* \right| \quad \text{then}$$

$$q_{V_i} + \left(\frac{\bar{B}\,\bar{\rho}}{\bar{\mu}} \right) \frac{\partial q_{V_i}}{\partial t} = -\frac{k_{ij}}{\bar{\mu}} \left(\frac{\partial \bar{p}}{\partial x_j} + \bar{\rho} g \frac{\partial z}{\partial x_j} \right) \rightarrow (12)$$

where

$$q_i \rightarrow \text{Specific discharge vector}$$

For laminar flows in porous media (also slow creeping motion), the local acceleration $\frac{\bar{B}\bar{\rho}}{\bar{\mu}}\frac{\partial q_i}{\partial t} \approx 0$

$\therefore$ We will get:

$$\boxed{q_i = -\frac{k_{ij}}{\bar{\mu}}\left(\frac{\partial \bar{p}}{\partial x_j} + \bar{\rho} g \frac{\partial z}{\partial x_j}\right)} \rightarrow (13)$$

Note:

$k_{ij} \rightarrow$ Permeability tensor

$\bar{p} \rightarrow$ average of pressure in fluid over REV

$\bar{\rho} \rightarrow$ average of density of entire fluid in REV

For homogeneous incompressible fluid, you will have

$\bar{\rho} = \rho =$ a constant

$\bar{\mu} = \mu =$ a constant

$$\therefore \boxed{q_i = \frac{-k_{ij}\rho g}{\mu}\frac{\partial}{\partial x_j}\left(\frac{\bar{p}}{\rho g} + z\right)}$$

or

$$\boxed{\vec{q} = -\frac{\underline{\underline{k}}\,\rho g}{\mu}\nabla \cdot \left(\frac{\bar{p}}{\rho g} + z\right)}$$

$\rightarrow (14)$

Equation (14) is the commonly used equation for specific discharge of a homogeneous, incompressible liquid in a porous medium.

You may also see that the term

$$\frac{k_{ij}\,\rho g}{\mu} = K_{ij}$$

↓

The hydraulic conductivity tensor.

Also

$$\frac{\bar{p}}{\rho g} + z = \bar{\Phi}$$

→ The piezometric head

$$\therefore \boxed{q_i = -K_{ij}\,\frac{\partial \bar{\Phi}}{\partial x_j}}$$

Quiz-2

Derive the conservation mas