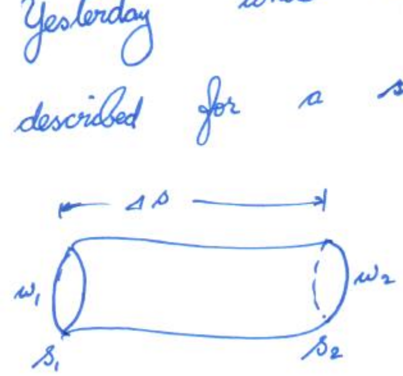
LECTURE 14

07-FEB-2014

Volume and Mass Conservation Equations for fluid in Porous Media

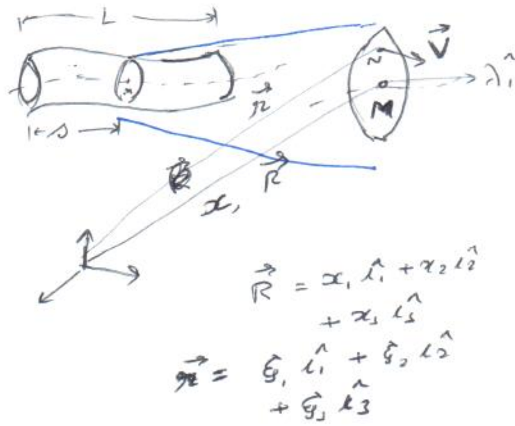
Yesterday while explaining volume conservation, we described for a stretch of channel void space in the REV



→ The local volume conservation equation for an incompressible liquid (~~or~~ fluid) is given as:

$$\boxed{\left\langle \frac{\partial V_s'}{\partial s} \right\rangle = 0} \rightarrow (1)$$

s → is the distance along channel axis



In the cross section area

→ channel of axis passes through M.

→ At arbitrary point N in the cross-section, the streamline direction is marked.

→ Using fixed co-ordinates $\vec{R} = x_1 \hat{i}_1 + x_2 \hat{i}_2 + x_3 \hat{i}_3$

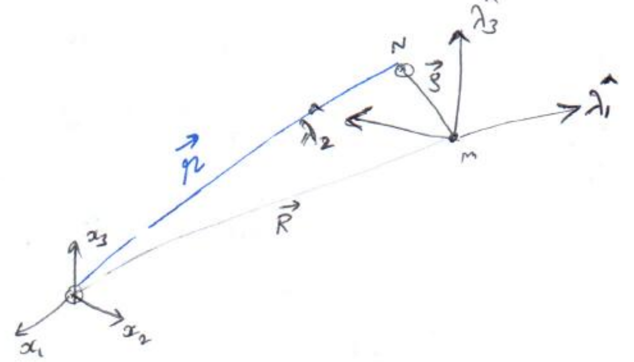
position vector of point M.

$$\vec{r} = \xi_1 \hat{i}_1 + \xi_2 \hat{i}_2 + \xi_3 \hat{i}_3$$

Position vector of point N.

As $\dot{s}$ is local, we can convert

$\left\langle \frac{\partial V_s'}{\partial s} \right\rangle = 0$ in the form of fixed co-ordinate system



$\vec{R} \rightarrow x_i$
$\vec{r} \rightarrow \xi_i$

$$\xi_i = x_i(s) + |\lambda_j| \cos(\hat{\lambda}_j, \hat{x}_i)$$

(This is index notation).

Also recall $\vec{\xi}_i = \vec{\xi}_i [\sigma(s)]$

From Bear & Bachmat we also have for entire REV,

$$\overline{\left(\frac{\partial g_\alpha}{\partial \xi_j}\right)} = \frac{\partial}{\partial x_j}\bar{g}_\alpha + \left(\bar{g}_\alpha - \bar{\tilde{g}}_\alpha\right) \frac{\partial}{\partial x_j}\left[\ln(n \Delta U_0)\right] \rightarrow (2)$$

where $\bar{g}_\alpha$ → Average of g_α over entire REV
$\bar{\tilde{g}}_\alpha$ → Average of $\tilde{g}_\alpha$ over entire circumferential lengths of channels in REV

If the porous media is considered as non-deforming, then porosity (n) will be constant.

Then we will have:

$$\overline{\left(\frac{\partial g_\alpha}{\partial t}\right)} \rightarrow$$ Average of temporal gradient of the property g_α

as:

$$\overline{\left(\frac{\partial g_\alpha}{\partial t}\right)} = \frac{\partial}{\partial t}\overline{g_\alpha}$$

This need not be the case everytime as there are chances porosity may vary

Also in equation 2 put $\bar{g}_\alpha = \overline{V_j'}$

and $\bar{\tilde{g}}_\alpha = 0$.

We will get

$$\overline{\left(\frac{\partial V_j'}{\partial \xi_j}\right)} = \frac{\partial \overline{V_j'}}{\partial x_j} + \left(\overline{V_j'} - 0\right)\frac{1}{n\Delta U_0}\frac{\partial (n\Delta U_0)}{\partial x_j}$$

For incompressible fluid the quantity $\nabla . \vec{V}' = 0$

$$\therefore \frac{\partial V_j'}{\partial \xi_j} = 0$$

$\therefore$ We have:

$$\frac{\partial \overline{V_j'}}{\partial x_j} + \frac{V_j'}{n\Delta U_0}\frac{\partial (n\Delta U_0)}{\partial x_j} = 0 \rightarrow (3)$$

Equation 3 can be written as

$$(n \Delta U_0) \frac{\partial}{\partial x_j} \bar{V}_j' + \bar{V}_j' \frac{\partial}{\partial x_j} (n \Delta U_0) = 0$$

Assuming $\Delta U_0 \approx$ constant

$$n \frac{\partial \bar{V}_j'}{\partial x_j} + \bar{V}_j' \frac{\partial n}{\partial x_j} = 0$$

or

$$\boxed{\frac{\partial}{\partial x_j} (n \bar{V}_j') = 0} \rightarrow 4$$

→ Equation 4 is the volume conservation equation in the entire form for an incompressible liquid in a porous media

→ The quantity $n \bar{V}_j'$ or $n \vec{V}'$ is also called specific discharge $\vec{q}$

or

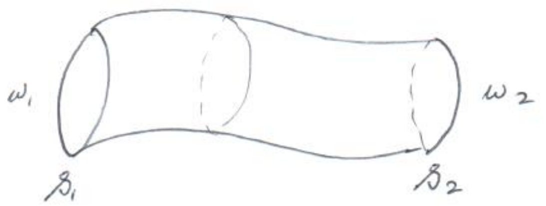
$$\boxed{\frac{\partial}{\partial x_j} q_j = 0}$$

Mass Conservation for Species α

Recall in a fluid continuum, the mass conservation equation for a species α was given as:

$$\left.\begin{aligned} \frac{\partial \rho_\alpha}{\partial t} + \nabla\cdot(\rho_\alpha \vec{V}_\alpha) &= I_\alpha \\ \text{or} \quad \frac{\partial \rho_\alpha}{\partial t} + \frac{\partial}{\partial x_j}(\rho_\alpha V_{\alpha_j}) &= I_\alpha \end{aligned}\right\} \rightarrow (5)$$

⇒ In a channel stretch of volume U_v in porous medium, we need to integrate the equation (5) in the volume U_v using the averaged quantities from the cross sectional area



Equation (5) averaged at any cross section and integrated over channel volume and then divided by REV.

$$\text{i.e.} \int_{s_1}^{s_2} \left\langle \frac{\partial \rho_\alpha}{\partial t} \right\rangle \omega \, ds + \int_{s_1}^{s_2} \left\langle \nabla\cdot \rho_\alpha \vec{V}_\alpha \right\rangle \omega \, ds = \int_{s_1}^{s_2} \langle I_\alpha \rangle \omega \, ds \quad \searrow (6)$$

$$\text{Now} \int_{s_1}^{s_2} \langle \nabla\cdot \rho_\alpha \vec{V}_\alpha \rangle \omega \, ds = \int_{\omega_1+\omega_2+S} \langle \rho_\alpha \vec{V}_\alpha \cdot \hat{n} \rangle \, dS$$

→ The channel axis is in s-direction

Velocity at any streamline $\vec{V}_\alpha$

∴ Projection of velocity along channel axis

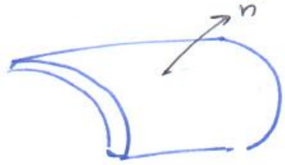
$$= \vec{V}_\alpha \cdot \hat{\lambda}_1$$

$$= V_{\alpha s}$$

→ Hence equation 6 will be written as:

$$\int_{s_1}^{s_2} \left\langle \frac{\partial \rho_\alpha}{\partial t} \right\rangle w \, ds = \left[\langle \rho_\alpha V_{\alpha s} \rangle w \right]_1 - \left[\langle \rho_\alpha V_{\alpha s} \rangle w \right]_2$$

$$- \int_S \langle \rho_\alpha \vec{V}_\alpha \cdot \hat{n} \rangle \, dS + \int_{s_1}^{s_2} \langle I_\alpha \rangle w \, ds$$



→ The average mass flux that crosses the solid boundaries is assumed as zero here.

i.e. $\int_S \langle \rho_\alpha \vec{V}_\alpha \cdot \hat{n} \rangle \, dS = 0$

Again assuming $I_\alpha = 0$, we get

$$\int_{s_1}^{s_2} \left\langle \frac{\partial \rho_\alpha}{\partial t} \right\rangle w \, ds = \left[\langle \rho_\alpha V_{\alpha s} \rangle w \right]_1 - \left[\langle \rho_\alpha V_{\alpha s} \rangle w \right]_2$$

~~In~~ limiting $\Delta s \to 0$ and dividing by $w \Delta s$, we get

$$\frac{1}{w \Delta s} \left\langle \frac{\partial \rho_\alpha}{\partial t} \right\rangle w \Delta s = -\frac{1}{w} \frac{\partial}{\partial s} \left[\langle \rho_\alpha V_{\alpha s} \rangle w \right]$$

$\therefore$ We get

$$\left\langle \frac{\partial \rho_\alpha}{\partial t} \right\rangle = -\left\langle \frac{\partial}{\partial s}(\rho_\alpha V_{\alpha s}) \right\rangle \qquad (\text{How ??})$$

So as $\Delta s \to 0$ we were getting

$$\left\langle \frac{\partial \rho_\alpha}{\partial t} \right\rangle = -\frac{1}{w}\frac{\partial}{\partial s}\left[\langle \rho_\alpha V_{\alpha s} \rangle w \right] \rightarrow (7)$$

Recall in equation

$$\frac{\partial}{\partial s}\langle g_\alpha \rangle \cong \left\langle \frac{\partial g_\alpha}{\partial s} \right\rangle + \left(\tilde{g}_\alpha - \langle g_\alpha \rangle \right) \frac{\partial w}{w\, \partial s}$$

if we put $g_\alpha = \rho_\alpha V_{\alpha s}$

and $\tilde{g}_\alpha = 0$, then

$$\frac{\partial}{\partial s}\langle \rho_\alpha V_{\alpha s} \rangle \cong \left\langle \frac{\partial}{\partial s}(\rho_\alpha V_{\alpha s}) \right\rangle - \langle \rho_\alpha V_{\alpha s} \rangle \frac{\partial w}{w\, \partial s}$$

$$\text{or} \quad \left\langle \frac{\partial}{\partial s}(\rho_\alpha V_{\alpha s}) \right\rangle = \frac{\partial}{\partial s}\langle \rho_\alpha V_{\alpha s} \rangle + \langle \rho_\alpha V_{\alpha s} \rangle \frac{\partial w}{w\, \partial s} \rightarrow (8)$$

The equation (7) on expansion gives:

$$\left\langle \frac{\partial \rho_\alpha}{\partial t} \right\rangle = -\frac{1}{w}\langle \rho_\alpha V_{\alpha s} \rangle \frac{\partial w}{\partial s} - \frac{\partial}{\partial s}\langle \rho_\alpha V_{\alpha s} \rangle$$

From (8) we will get

$$\left\langle \frac{\partial \rho_\alpha}{\partial t} \right\rangle = -\left\langle \frac{\partial}{\partial s}(\rho_\alpha V_{\alpha s}) \right\rangle \longrightarrow (9)$$

Now recall about diffusive mass flux

$$\vec{J}_\alpha^* = \rho_\alpha (\vec{V}_\alpha - \vec{V}^*)$$

or $$\vec{J}_\alpha' = (\rho_\alpha (\vec{V}_\alpha - \vec{V}'))$$

Lightfoot and Cussler (1965) suggested linear relation between mass (or property) movement and density gradients for diffusive fluxes

$$\vec{J}_\alpha' = \rho_\alpha (\vec{V}_\alpha - \vec{V}') = -\sum_{\beta=1}^{N} D_{\alpha\beta} \nabla \rho_\beta$$

$\vec{J}_\alpha' \rightarrow$ Mass flux

$D_{\alpha\beta} \rightarrow$ a coefficient called Molecular Diffusivity

Now if we suggest:

$$\rho_\alpha V_{\alpha s} = \rho_\alpha (V_{\alpha s} - V_s') + \rho_\alpha V_s'$$

i.e. $$\rho_\alpha V_{\alpha s} = -D_{\alpha\beta} \nabla \rho_\alpha + \rho_\alpha V_s'$$

For a binary system, i.e two specie fluid

$$\nabla \rho_\alpha = \frac{\partial \rho_\alpha}{\partial \sigma} \frac{d\sigma}{ds}$$

=> The equation for mass conservation for α-specie is (locally averaged)

$$\boxed{\left\langle \frac{\partial \rho_\alpha}{\partial t} \right\rangle = \left\langle \frac{\partial}{\partial \xi_i} \left(D_{\alpha\beta} T_{ij}^* \frac{\partial \rho_\alpha}{\partial \xi_j} \right) \right\rangle - \left\langle \frac{\partial}{\partial \xi_i} (\rho_\alpha V_i') \right\rangle}$$

↳ 10

where

$$V_i' = V \frac{d\xi_i}{d\sigma}$$

$$T_{ij}^* = T_{ij} \left(\frac{d\sigma}{ds}\right)^2$$

$$T_{ij} = \frac{d\xi_i}{d\sigma} \frac{d\xi_j}{d\sigma}$$

This is also media's tortuosity

Tensor whose elements are products of cosines of angles between streamline direction at a point and the coordinate axis.

The equation

$$\boxed{\left\langle \frac{\partial \rho_\alpha}{\partial t} \right\rangle = \frac{\partial}{\partial \xi_i} \left(D_{\alpha\beta} T_{ij}^* \frac{\partial \rho_\alpha}{\partial \xi_j} \right) - \left\langle \frac{\partial}{\partial \xi} (\rho_\alpha V_i') \right\rangle} \rightarrow (11)$$

⇒ This is averaged over entire void space.

ΔU_0 = constant.

$$\frac{\partial(\bar{\rho}_\alpha)}{\partial t} = \frac{\partial}{\partial x_i} D_{\alpha\beta} \overline{T}_{ij}^* \frac{\partial \rho_\alpha}{\partial x_j} - \frac{\partial}{\partial x_i} \overline{\rho_\alpha V_i'} - \frac{\overline{\rho_\alpha V_i'}}{n} \frac{\partial n}{\partial x_i}$$

This expression • $\rho_\alpha V_i'$ → instantaneous mass flux of the α- species carried by the liquid

This instantaneous mass flux consists of mass flux carried by average fluid motion in REV (i.e $\rho_\alpha \bar{V}_i'$)

and dispersive flux $\left(\rho_\alpha \mathring{V}_i' \right)$ due to the velocity fluctuations.

$\Rightarrow$ Extending the knowledge, for entire fluid the mass <u>conservation equation in REV</u> can be given as:

$$\boxed{\frac{\partial}{\partial t}\bar{\rho} = \frac{\partial}{\partial x_i}\left[\left(D_{ij} + D_{\alpha\beta}\,\overline{T}^*_{ij}\right)\frac{\partial \bar{\rho}}{\partial x_j}\right] - \frac{\partial}{\partial x_i}\left(\bar{\rho}\,\bar{V}_i'\right)} \longrightarrow (12)$$

where D_{ij} $\rightarrow$ ~~Hydrodynamic~~ Mechanical dispersion tensor

$D_{\alpha\beta}$ $\rightarrow$ Molecular diffusivity

$\overline{T}^*_{ij}$ $\rightarrow$ Media's tortuosity

$\bar{\rho}$ $\rightarrow$ Average of density over entire REV

$\bar{V}_i'$ $\rightarrow$ Component of volume averaged velocity over REV in i- direction.

Note:- While doing local averaging in conservation of mass for specie α, we had arrived at

$$\left\langle \frac{\partial \rho_\alpha}{\partial t} \right\rangle = -\left\langle \frac{\partial}{\partial s}\left(\rho_\alpha V_{\alpha s}\right)\right\rangle$$

Also $\rho_\alpha V_{\alpha s} = -D_{\alpha\beta}\nabla\rho_\alpha + \rho_\alpha V_s'$

$$\therefore \left\langle \frac{\partial \rho_\alpha}{\partial t} \right\rangle = -\left\langle \frac{\partial}{\partial s}\left(\rho_\alpha V_s'\right)\right\rangle + \left\langle \frac{\partial}{\partial s}\left(D_{\alpha\beta}\nabla\rho_\alpha\right)\right\rangle$$

Also $\nabla\rho_\alpha = \dfrac{\partial \rho_\alpha}{\partial \sigma}\dfrac{d\sigma}{ds}$;

As s is along channel axis, if we can represent spatial partial derivatives w.r.t the fixed co-ordinates then recall

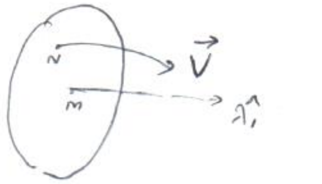
$$\hat{\xi}_i = \hat{\xi}_i[\sigma(s)]$$

(i.e Position vector $\vec{r} = \xi_1 \hat{i}_1 + \xi_2 \hat{i}_2 + \xi_3 \hat{i}_3$ is function of σ, which itself is function of 's').

$$\therefore \quad \frac{\partial}{\partial s}(\ ) = \frac{\partial(\ )}{\partial \xi_j} \frac{d\xi_j}{d\sigma} \frac{d\sigma}{ds}$$

So this property is utilised in equation (9).

Also



$$V_s = \vec{V} \cdot \hat{\lambda}_1$$
$$= |V| \, \hat{\sigma} \cdot \hat{\lambda}_1$$
$$= |V| \frac{d\xi_i}{d\sigma} \cdot \frac{dx_i(s)}{ds}$$

Using relations between ξ and x,

we have
$$V_s \approx \frac{V}{(d\sigma/ds)}$$

$$\therefore \quad V_s' = \frac{V'}{(d\sigma/ds)} \quad \left(\text{for volumetric-averaged velocity.}\right)$$

Substituting in (9)

$$\left\langle \frac{\partial \rho_\alpha}{\partial t} \right\rangle = -\left\langle \frac{\partial}{\partial \xi_i}\left(\rho_\alpha \frac{V'}{d\sigma/ds}\right) \frac{d\xi_i}{d\sigma} \frac{d\sigma}{ds} \right\rangle$$
$$+ \left\langle \frac{\partial}{\partial \xi_i}\left(D_{\alpha\beta} \frac{\partial \rho_\alpha}{\partial \xi_j} \frac{d\xi_j}{d\sigma} \frac{d\sigma}{ds}\right) \frac{d\xi_i}{d\sigma} \frac{d\sigma}{ds} \right\rangle$$

$\nabla \rho_\alpha$ in fixed coordinates

$$\frac{\partial \rho_\alpha}{\partial \xi_j} \frac{d\xi_j}{d\sigma} \frac{d\sigma}{ds}$$

i.e. ~~$\left\langle \frac{\partial \rho_\alpha}{\partial t} \right\rangle$~~ Put $V_i' = V \frac{d\xi_i}{d\sigma}$

∴ We will get:

$$\left\langle \frac{\partial \rho_\alpha}{\partial t} \right\rangle = -\left\langle \frac{\partial}{\partial \xi_i}(\rho_\alpha V_i') \right\rangle + \left\langle \frac{\partial}{\partial \xi_i}\left(D_{\alpha\beta} T_{ij}^{*} \frac{\partial \rho_\alpha}{\partial \xi_j} \right) \right\rangle$$

where $T_{ij}^{*} = T_{ij}\left(\frac{d\sigma}{ds}\right)^2$; $T_{ij} = \frac{d\xi_i}{d\sigma} \cdot \frac{d\xi_j}{d\sigma}$

For REV averaged quantity, we have.

$$\frac{\partial}{\partial t}\bar{\rho}_\alpha = \frac{\partial}{\partial x_i}\left(D_{\alpha\beta} \overline{T_{ij}^{*}} \frac{\partial \bar{\rho}_\alpha}{\partial x_j} \right) - \frac{\partial}{\partial x_i} \overline{\rho_\alpha V_i'} - \frac{\overline{\rho_\alpha V_i'}}{n} \frac{\partial n}{\partial x_i}$$

where $\rho_\alpha V_i' = \rho_\alpha \bar{V}_i' + \rho_\alpha \mathring{V}_i'$

where $\rho_\alpha \mathring{V}_i' \rightarrow$ dispersive flux resulting from fluctuations in velocity.

Now for entire REV ; $\overline{\rho_\alpha \mathring{V}_i'} \approx -D_{ij} \frac{\partial \bar{\rho}_\alpha}{\partial x_j}$

where $D_{ij} \rightarrow$ Mechanical dispersion tensor

∴ REV averaged mass conservation:

$$\frac{\partial}{\partial t}\bar{\rho}_\alpha = \frac{\partial}{\partial x_i}\left[\left(D_{ij} + D_{\alpha\beta} \overline{T_{ij}^{*}} \right) \frac{\partial \bar{\rho}_\alpha}{\partial x_j} \right] - \frac{\partial}{\partial x_i}(\bar{\rho}_\alpha \bar{V}_i')$$

(Ofcourse by assuming n = constant)