LECTURE 13
06-FEB-2014

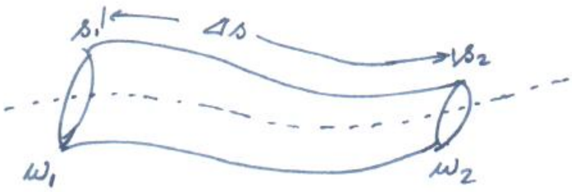
Volume and Mass Conservation In Porous Media

In the last class we discussed on averaging the properties (say g_α)

* First, over a cross sectional area of a channel void space, i.e. $\langle g_\alpha \rangle$
* Second, over the entire void space channels in the REV, i.e. $\bar{g}_\alpha$

$\Rightarrow$ The first average gives local transport equation

Volume Conservation



Consider a small length of a channel void space between s_1 and s_2

Length of this portion $= \Delta s$

Area of cross-section at $s_1 = \omega_1$
" " " at $s_2 = \omega_2$

$\therefore$ Volume of void space in this channel stretch $= U_v$

i.e. $U_v = \int_{s_1}^{s_2} \omega(s)\, ds$

Recall the conservation equation :

$$\frac{\partial g_\alpha}{\partial t} + \nabla \cdot (g_\alpha \vec{V}_{G\alpha}) = I_\alpha$$

If $g_\alpha = \rho$, density of entire fluid
$\vec{V}_{\alpha\alpha} = \vec{V}'$ volume averaged velocity
Then mass conservation is given by :

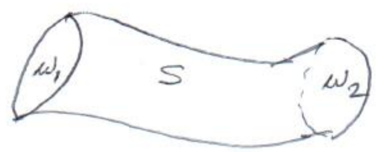
$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{V}') = 0$$

For an incompressible liquid : ρ = constant

$$\therefore \boxed{\nabla \cdot \vec{V}' = 0} \rightarrow (1)$$

Again equation (1) applied over ~~and~~ void volume in channel between stretch s_1 and s_2

$$\int_{U_v} (\nabla \cdot \vec{V}')\, dU_v = 0$$

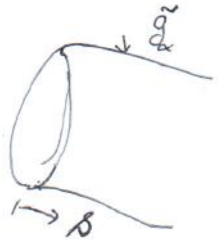


Using Gauss divergence theorem

$$\int_{U_v} (\nabla \cdot \vec{V}')\, dU_v = \int_{(\omega_1 + \omega_2 + S)} (\vec{V}' \cdot \hat{n})\, dS = 0 \rightarrow (2)$$

To further proceed, let us recall the local averaging for any density of a property (i.e. g_α)

→ The average over a cross sectional area $\langle g_\alpha \rangle$



→ Let $\tilde{g}_\alpha$ be average of this density on the circumference of the channel, such that $\qquad |g_\alpha - \tilde{g}_\alpha| << \tilde{g}_\alpha$.

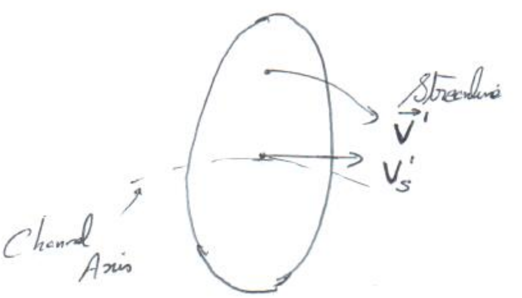
Bear and Bachmat (1966, 67) suggested that we can then suggest the differential for that property as such:

$$\frac{\partial}{\partial s}\langle g_\alpha\rangle \cong \left\langle \frac{\partial g_\alpha}{\partial s}\right\rangle + \left(\tilde{g}_\alpha - \langle g_\alpha\rangle\right)\frac{\partial \omega}{\omega\,\partial s} \rightarrow 3$$

{Change in average of g_α per unit length of channel} $\qquad$ {Average of change in g_α per unit length of channel}

Now equation 2 for incompressible liquid is written as:

$$\int_{\omega_1+\omega_2} (\vec{V}'\cdot\hat{n})\, dS = 0$$



Assuming the component of velocity across the channel circumference is zero.

i.e. $\quad \oint \left[\langle V_s'\rangle\,\omega\right]_{s_2} - \left[\langle V_s'\rangle\,\omega\right]_{s_1} = 0$

If $\Delta s \rightarrow 0$, $\quad \underset{\Delta s\rightarrow 0}{Lt}\ \dfrac{\left[\langle V_s'\rangle\,\omega\right]_{s_2} - \left[\langle V_s'\rangle\,\omega\right]_{s_1}}{\Delta s} = 0$

i.e. $\quad \dfrac{\partial}{\partial s}\left(\langle V_s'\rangle\,\omega\right) = 0$

or $\quad \omega\,\dfrac{\partial}{\partial s}\langle V_s'\rangle + \langle V_s'\rangle\,\dfrac{\partial \omega}{\partial s} = 0 \quad \rightarrow 4$

Put $\quad g_\alpha = V_s'$ in equation 3 and compare with 4

We will get (Note: $\tilde{g}_\alpha = V_s'$ on circumference $= 0$)

$$\left\langle \frac{\partial V_s'}{\partial s} \right\rangle = 0 \longrightarrow (5)$$

This is local equation of fluid volume conservation.

Again from analogy of equation (3), extending it into entire REV,

Let $\bar{g}_\alpha \rightarrow$ average of density over REV

$\bar{\tilde{g}}_\alpha \rightarrow$ average of density on the circumference of various channel void space in the REV

Considering the porous medium as non-deforming (i.e $n =$ porosity $=$ a constant), we have:

$$\overline{\left(\frac{\partial g_\alpha}{\partial t} \right)} = \frac{\partial}{\partial t}(\bar{g}_\alpha)$$

Average of gradient of density w.r.t unit time

Gradient of average density per unit time

IIIly as $\vec{R}$ is to be used for any channel void location we can write:

$$\overline{\left(\frac{\partial g_\alpha}{\partial \xi_j} \right)} = \frac{\partial \bar{g}_\alpha}{\partial x_j} + \left(\bar{g}_\alpha - \bar{\tilde{g}}_\alpha \right) \frac{\partial}{\partial x_j}\left(\ln (n \Delta U_0) \right) \longrightarrow (6)$$

Here putting $\bar{g}_\alpha = \bar{V}_j'$, $\bar{\tilde{g}}_\alpha = 0$