TRANSPORT EQUATIONS ....

Last day in the quiz, we asked you to derive the conservation of mass of species - α in a multi - component fluid using mass average velocity $\vec{V}^*$. The general equation for conservation is

$$\frac{\partial g_\alpha}{\partial t} + \nabla \cdot \left(g_\alpha \vec{V}_{G\alpha} \right) = I_\alpha \rightarrow (1)$$

Looking at your responses, I am now again providing the derivation briefly.

For a specie - α in a multi-specie fluid let $g_\alpha = \rho_\alpha$ and $\vec{V}_{G\alpha} = \vec{V}_\alpha$

$\therefore$ (1) becomes

$$\frac{\partial \rho_\alpha}{\partial t} + \nabla \cdot \left(\rho_\alpha \vec{V}_\alpha \right) = I_\alpha \rightarrow (2)$$

or

$$\frac{\partial \rho_\alpha}{\partial t} + \rho_\alpha \left(\nabla \cdot \vec{V}_\alpha \right) + \left(\vec{V}_\alpha \cdot \nabla \right) \rho_\alpha = I_\alpha \rightarrow (2A)$$

Now again using material derivative concept recall that for any property B_α of an G_α particle (fluid),

$$\frac{D B_\alpha}{D t} = \frac{\partial B_\alpha}{\partial t} + \left(\vec{V}_{G\alpha} \cdot \nabla \right) B_\alpha \rightarrow (3)$$

In equation 3 now put ~~b~~ $B_\alpha = \rho_\alpha$ and $\vec{V}_{a\alpha} = \vec{V}^*$

$\therefore$ We can now write:

$$\frac{D^*\rho_\alpha}{Dt} = \frac{\partial \rho_\alpha}{\partial t} + (\vec{V}^* \cdot \nabla)\rho_\alpha \longrightarrow (4)$$

Utilising eqn. 4 in 2A we get:

$$\frac{D^*\rho_\alpha}{Dt} + (\vec{V}_\alpha - \vec{V}^*) \cdot \nabla \rho_\alpha + \rho_\alpha \nabla \cdot \vec{V}_\alpha = I_\alpha \rightarrow (5)$$

Recall diffusive velocities and diffusive fluxes (based on mass average velocity)

$\hat{\vec{V}}_\alpha^* = \vec{V}_\alpha - \vec{V}^*$ and $\vec{J}_\alpha^* = \rho_\alpha (\vec{V}_\alpha - \vec{V}^*)$

Divergence of diffusive mass flux:

$$\nabla \cdot \vec{J}_\alpha^* = \rho_\alpha \nabla \cdot (\vec{V}_\alpha - \vec{V}^*) + (\vec{V}_\alpha - \vec{V}^*) \cdot \nabla \rho_\alpha$$

or $$(\vec{V}_\alpha - \vec{V}^*) \cdot \nabla \rho_\alpha = \nabla \cdot \vec{J}_\alpha^* - \rho_\alpha \nabla \cdot (\vec{V}_\alpha - \vec{V}^*) \longrightarrow (6)$$

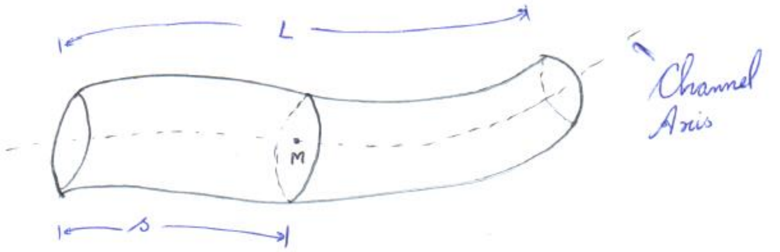
Substituting 6 in 5:

$$\frac{\partial \rho_\alpha}{\partial t} + (\vec{V}^* \cdot \nabla)\rho_\alpha + \nabla \cdot \vec{J}_\alpha^* - \rho_\alpha \nabla \cdot (\vec{V}_\alpha - \vec{V}^*) + \rho_\alpha(\nabla \cdot \vec{V}_\alpha) = I_\alpha$$

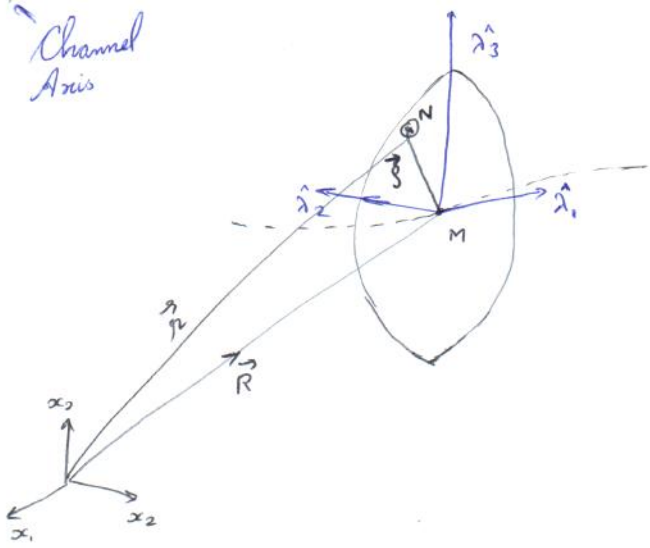
i.e. $$\boxed{\frac{\partial \rho_\alpha}{\partial t} + \underbrace{\nabla \cdot (\rho_\alpha \vec{V}^*)}_{\text{Advective mass flux}} + \underbrace{\nabla \cdot \vec{J}_\alpha^*}_{\text{Diffusive mass flux}} = I_\alpha} \rightarrow (7)$$

Now recalling last day's lecture about porous medium model consisting of channels and junctions.

A Channel :



$0 \leq s \leq L$



$\vec{R} = \vec{R}(s)$

or $\quad x_i = x_i(s)$

→ The position of point M that is on channel axis is given by position vector $\vec{R}$ (or say x_i).

→ There ~~are~~ is another co-ordinate system at point M having unit vectors $\hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_3$.

→ The point N lies on the plane $\lambda_2 \lambda_3$.

The position to N is given by $\vec{r}$

$$\vec{r} = \vec{R} + \vec{\xi}$$

where $\vec{\xi}$ → radius vector of N in $\lambda_2 \lambda_3$ plane

→ This position vector $\vec{r}$ is now local coordinate for N and in index notation can be given as $\vec{\xi}_i$; $i = 1, 2, 3$

$$\vec{\xi}_i = x_i(s) + \rho_j(\lambda_j, x_i) \quad ; \quad j = 2, 3$$

In ~~laminar~~ fluid flow streamlines will not intersect.
Each such point N on the cross sectional area will allow one streamline through it.

$\Rightarrow$ The origin of this streamline can be associated to the beginning of the channel

$\therefore$ At $s = 0$, streamlines begin.

Let us suggest 'σ' $\rightarrow$ the distance moved along the streamline.

$\therefore$ At $s = 0$, $\sigma = 0$

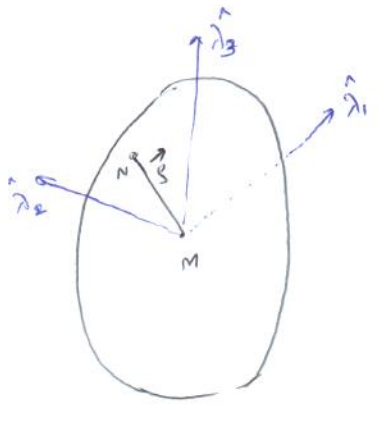
$\therefore$ We can also now consider

$$\vec{\xi}_i = \vec{\xi}_i(\sigma(s))$$

$\rightarrow$ The Averaging Procedure :

(i) Averaging the property over the entire cross sectional area in a channel.

(ii) Averaging this averaged property to the entire void spaces in the REV.



Considering the cross sectional area $\omega(s)$ at the channel axial point M, any point N will allow a streamline to pass through it.

The streamline passing through N will be at a distance σ from channel origin.

$\therefore$ Let us assume $g_\alpha(\sigma)$ a <u>scalar property</u> inside this void space.

The first average suggest we need to average this property to the entire cross section

i.e. $$\langle g_\alpha(s) \rangle = \frac{1}{\omega(s)} \int_{[\omega(s)]} g_\alpha(\sigma)\, d\omega(\sigma)$$

$$= \frac{1}{\omega(s)} \int_{\omega(s)} g_\alpha(s, \lambda_2, \lambda_3)\, d\lambda_2\, d\lambda_3$$

i.e $d\omega = d\lambda_2\, d\lambda_3$

If g_α is differentiable inside the void space,

then $$\langle \nabla g_\alpha \rangle = \left\langle \frac{\partial g}{\partial s} \right\rangle \hat{\lambda}_1 + \left\langle \frac{\partial g}{\partial \lambda_2} \right\rangle \hat{\lambda}_2 + \left\langle \frac{\partial g}{\partial \lambda_3} \right\rangle \hat{\lambda}_3$$

If g_α is $\vec{V}$, then $\vec{V} = V_s \hat{\lambda}_1 + V_{\lambda_2} \hat{\lambda}_2 + V_{\lambda_3} \hat{\lambda}_3$

The second averaging involves averaging this property over entire void space in REV.

⇒ If volume of REV is ΔU_0 and porosity is 'n'.

⇒ Then volume of voids $= n \Delta U_0$

G_α → the extensive fluid property based on density g_α

is

$$G_\alpha = \int_{(n\Delta U_0)} g_\alpha \, dU_v = \int_{\Delta U_{cha}} g_\alpha \, dU_{cha} + \int_{\Delta U_{junct}} g_\alpha \, dU_{junct}$$

(Volume of voids consists of volumes of channels and junctions).

As volume of junctions are neglected:

$$G_\alpha = \int_{\Delta U_{channel}} g_\alpha \, dU_{channel} = \sum_{j=1}^{M} \int_0^{L_p} \langle g_\alpha \rangle \, \omega(s) \, ds$$

Also:

$$\bar{g}_\alpha (P) = \frac{1}{(n \Delta U_0)_P} \int_{(n\Delta U_0)_P} g_\alpha \, dU_v$$