

### St. VENANT's MOMENTUM EQUATION

Yesterday, we started discussing the derivation of St. Venant's momentum equation.

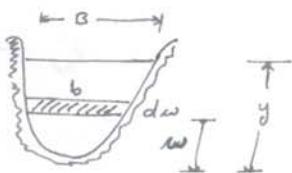
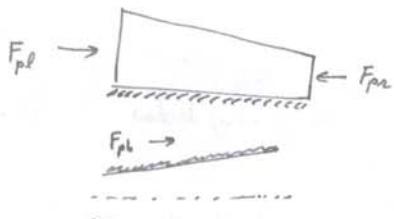


$$\vec{F} = \frac{d}{dt} \iiint_{cv} \vec{v} s dV + \iint_{cs} \vec{v} s \vec{v} \cdot \vec{n} dA$$

The various forces acting on the control volume are:

- \* Gravity force  $F_g$
- \* Frictional force  $F_f$
- \* Contraction / Expansion,  $F_e$
- \* Wind shear force,  $F_w$
- \* Net pressure force,  $F_p$

Regarding the net pressure force



$$F_{pl} = \int_0^y \rho g (y - w) b dw$$

$$\text{By on right side } F_{pn} = -\left(F_{pl} + \frac{\partial F_{pl}}{\partial x} dx\right)$$

$$\text{Now } \frac{\partial F_{pl}}{\partial x} = \int_0^y \rho g \frac{\partial y}{\partial x} b dw + \int_0^y \rho g (y - w) \frac{\partial b}{\partial x} dw$$

$$\text{i.e. } \frac{\partial F_{pl}}{\partial x} = \rho g A \frac{\partial y}{\partial x} + \int_0^y \rho g (y - w) \frac{\partial b}{\partial x} dw$$

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- As suggested yesterday, the pressure will exert a force perpendicular and into the plane of consideration.
- The left and right boundaries are bounded by water and we have accommodated  $F_{pl}$  and  $F_{pr}$ .
- The front, back, and bottom of the control volume are bounded by banks and beds. If you isolate the control volume you need to take into account the pressure forces.
- There will be a component of this pressure force in the direction of flow. This is mainly due to change in width from left to right.

Please note the rate of change of width =  $\frac{\partial b}{\partial x}$

$$\therefore F_{pb} = \left[ \int_0^y sg(y-w) \frac{\partial b}{\partial x} dw \right] \Delta x$$

$$\begin{aligned}\therefore F_p &= F_{pl} + F_{pr} + F_{pb} \\ &= \int_0^y sg(y-w)b dw - \int_0^y sg(y-w)b dw \\ &\quad - \int_0^y sg \frac{\partial y}{\partial x} b dw \Delta x - \int_0^y sg(y-w) \frac{\partial b}{\partial x} dw \Delta x \\ &\quad + \int_0^y sg(y-w) \frac{\partial b}{\partial x} dw \Delta x\end{aligned}$$

$$\therefore \underline{F_p = -sg A \frac{\partial y}{\partial x} \Delta x}$$

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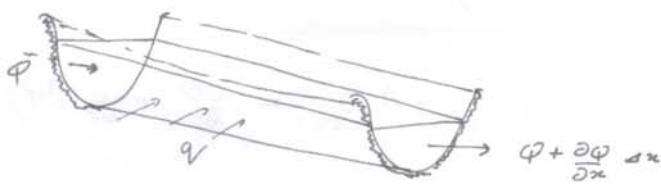
Hence the net force  $\vec{F}$  in the control volume

$$\vec{F} = \rho g A S_0 \Delta x - \rho g A S_1 \Delta x - \rho g A S_2 \Delta x - w_f B S \Delta x \\ - \rho g A \frac{\partial y}{\partial x} \Delta x$$

This is the LHS of the RTT for momentum

$$\vec{F} = \frac{d}{dt} \iiint_{cv} \vec{v} s dV + \iint_{cs} \vec{v} s \vec{v} \cdot \hat{n} dA$$

Now the net outflow of the momentum:



Let us consider  
v as the average  
velocity at the  
cross section having  
discharge Q.

$\therefore$  Momentum flux at that section =  $\rho \beta V Q$

$\beta \rightarrow$  Momentum correction factor =  $\frac{1}{V^2 A} \iint v^2 dA$

$$\therefore \iint_{cs} \vec{v} s \vec{v} \cdot \hat{n} dA = \rho \left[ \beta V Q + \frac{\partial}{\partial x} (\beta V Q) \Delta x \right] \\ - \rho \left[ \beta V Q + \beta v_n q \Delta x \right]$$

where  $\rho \beta v_n q \rightarrow$  Momentum flux entering through the lateral sides

$$\therefore \iint_{cs} \vec{v} s \vec{v} \cdot \hat{n} dA = \rho \left[ \frac{\partial}{\partial x} (\beta V Q) - \rho v_n q \right] \Delta x$$

$v_n \rightarrow$  component of lateral inflow velocity in x-direction.

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The change in momentum stored inside the cv:

$$\frac{d}{dt} \iiint_{cv} \vec{v} s du = \frac{\partial}{\partial t} (s A \Delta x v) = s \frac{\partial \varphi}{\partial t} \Delta x$$

$\therefore$  The momentum equation becomes:

$$sg A s_0 \Delta x - sg A s_g \Delta x - sg A s_e \Delta x - w_f B s \Delta x - sg A \frac{\partial y}{\partial x} \Delta x \\ = s \frac{\partial \varphi}{\partial t} \Delta x + s \left[ \frac{\partial}{\partial x} (\beta V \varphi) - \beta V_x q \right] \Delta x$$

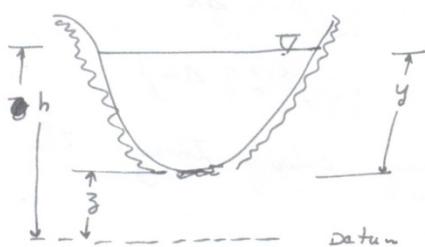
Looking into this expression, we can eliminate  $s \Delta x$  throughout and get

$$g A (s_0 - s_g - s_e) - w_f B - g A \frac{\partial y}{\partial x} = \frac{\partial \varphi}{\partial t} + \frac{\partial}{\partial x} \left( \frac{\beta \varphi^2}{A} \right) - \beta V_x q$$

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$$\text{i.e. } \frac{\partial \varphi}{\partial t} + \frac{\partial}{\partial x} \left( \frac{\beta \varphi^2}{A} \right) + g A \left( \frac{\partial y}{\partial x} - s_0 + s_g + s_e \right) + w_f B - \beta V_x q = 0$$

This is conservation form of momentum equation.



Now based on any datum

$$y = h - z$$

$$\therefore \frac{\partial y}{\partial x} = \frac{\partial h}{\partial x} - \frac{\partial z}{\partial x}$$

$$\text{But } \frac{\partial z}{\partial x} = -s_0$$

$$\therefore \frac{\partial y}{\partial x} = \frac{\partial h}{\partial x} + s_0$$

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$$\therefore \frac{\partial \varphi}{\partial t} + \frac{\partial}{\partial x} \left( \frac{\beta \varphi^2}{A} \right) + g A \left( \frac{\partial h}{\partial x} + s_g + s_e \right) + w_f B - \beta V_x q = 0$$

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You can also develop non-conservation form of momentum equation. We need to consider unit width of the channel and assume that the velocity of flow through that unit width is  $V$ .

Then  $A = y$ ,  $\sigma = Vy$ .

Also we need to neglect Eddy loss, wind shear, and lateral inflow. We will get:

$$\frac{\partial V}{\partial t} + V \frac{\partial V}{\partial x} + g \left( \frac{\partial y}{\partial x} - s_0 + \xi \right) = 0$$

### Classification of Distributed Routing

Neglecting lateral inflow, wind shear, and Eddy losses the Saint Venant's equations are

$$\frac{\partial Q}{\partial x} + \frac{\partial A}{\partial t} = 0$$

$$\frac{1}{A} \frac{\partial Q}{\partial t} + \frac{1}{A} \frac{\partial}{\partial x} \left( \frac{Q^2}{A} \right) + g \left( \frac{\partial y}{\partial x} - s_0 + \xi \right) = 0$$

Assuming momentum correction factor  $\beta \approx 1.0$

$\frac{1}{A} \frac{\partial Q}{\partial t}$  → This is the local acceleration term (i.e. change in momentum due to change in velocity over a period of time).

$\frac{1}{A} \frac{\partial}{\partial x} \left( \frac{Q^2}{A} \right)$  → Convective acceleration term. Change in momentum due to change in velocity along the length of the channel.

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$\rho \frac{\partial y}{\partial x}$  → the pressure force term.

$\rho g s_0$  → the gravity force term

$\rho g s_f$  → the friction force term.

The inertial forces on the flow are represented by local and convective accelerations.

The non-conservative form of momentum equation  
(i.e. for unit width of the element)

$$\frac{\partial V}{\partial t} + V \frac{\partial V}{\partial x} + \rho \frac{\partial y}{\partial x} - \rho (s_0 - s_f) = 0$$

All three Dynamic Wave Model.

→ Based on the number of terms you like to involve while routing, you may have different distributed models.

- \* If the distributed model does not account local and convective accelerations as well as pressure force, then only gravitational and friction forces are present

$$\text{i.e. } -\rho (s_0 - s_f) = 0$$

$$\text{or } s_0 = s_f$$

The friction force is balanced by gravity force.

Such models are kinematic wave models.

- \* If you also include pressure force terms along with gravity and friction. However acceleration terms are neglected. → Diffusion wave models.