17/02/2017

Lecture 16

(*Refer the text book "CONTINUUM MECHANICS" by GEORGE E. MASE, Schaum's Outlines)*

Kinematics of Fluids

- Last class, we started discussing about the kinematics of fluids.
- Recall the ***Lagrangian*** and ***Eulerian*** way of analyzing fluid motion.
- As the continuum encompasses of several particles, the Lagrangian analysis deals with each particles.

- To quickly describe certain quantities, consider a continuum in motion.
  Initially at time $t=t_o$, the slope of the continuum is as shown and is referred with the orthogonal coordinates $OX_1X_2X_3$.

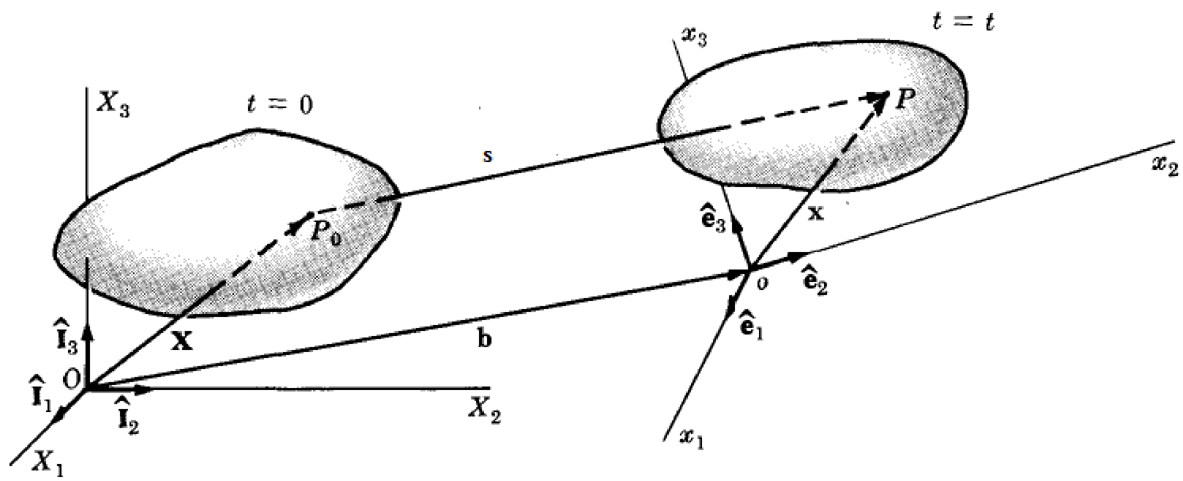

Fig. 1: Representation of the statement above
(Source: Schaum's outline of theory and problems of continuum mechanics by George Mase)

At a mathematical point P_0, a particle of the continuum is associated at time $t=t_0$.
The position vector of this particle is $\vec{X} = X_1\hat{I}_1 + X_2\hat{I}_2 + X_3\hat{I}_3$
The above position vector is called ***material coordinate***.

- As the continuum is moving, at a later time t=t, the position of the same particle might have changed.
- Even the continuum also deforms shifts its position.

Let the position at time ‘t’ be given as in the figure $\vec{x} = x_1\hat{e}_1 + x_2\hat{e}_2 + x_3\hat{e}_3$
This new position description is ***spatial coordinates.***

- The same particle that was present at P_0 at time t_0 is now displaced and the displacement vector is given as $\vec{s}$.

- You can also see that the coordinates also shifted by a vector $\vec{b}$.
  ➔ From vector algebra:
  $\vec{s} = \vec{b} + \vec{x} - \vec{X}$

- If the coordinates $OX_1X_2X_3$ and $ox_1x_2x_3$ are merged, you get $\vec{b} = 0$
  Hence, $\vec{s} = \vec{x} - \vec{X}$
  (This means, $\vec{x}$ is the position vector of the particle at time ‘t’ , whose initial position is $\vec{X}$ **).**
  In index notation: $s_k = x_k - X_k$

- When the continuum is in motion and deformation, the particles position may be expressed in the form:
  $x_i = x_i(X_1, X_2, X_3, t)$ or $\vec{x} = \vec{x}(\vec{X}, t)$
  You know, x_i → present location of the particle that occupied the point (X_1, X_2, X_3) at time $t = t_o$.
  (This is mapping the initial configuration with the current configuration). Such type of motion description is ***Lagrangian formulation.***

- The dependent quantity is x_i and independent quantity is X_i.

- If the motion or deformation is represented by the form:
  $X_i = X_i(x_1, x_2, x_3, t)$ or $\vec{X} = \vec{X}(\vec{x}, t)$
  where independent variable is x_i and t.
  This is ***Eulerian formulation.***
  This description provides you the tracing of original position of the particle that now occupies the spatial coordinate or location (x_1, x_2, x_3).

- The Lagrangian and Eulerian mappings are therefore inverse functions. For the inverse functions to exist, the necessary requirement is that the Jacobian ***must exist.***
  Jacobian:

$$J = \left|\frac{\partial x_i}{\partial X_j}\right| = \begin{vmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{vmatrix}$$

If this Jacobian (determinant) is zero, then unique inverse does not exist.

- From $x_i = x_i(X_1, X_2, X_3, t)$, → the Lagrangian form, you can form ***material deformation gradient*** by partially differentiating it with $\vec{X}$

i.e., $$\frac{\partial x_i}{\partial X_j} = \begin{pmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{pmatrix}$$ Note: This is not a determinant. It is a tensor

F_{ij}

- From $X_i = X_i(x_1, x_2, x_3, t)$ → The Eulerian form, you get spatial deformation gradient

$$\frac{\partial X_i}{\partial x_j} = \begin{pmatrix} \frac{\partial X_1}{\partial x_1} & \frac{\partial X_1}{\partial x_2} & \frac{\partial X_1}{\partial x_3} \\ \frac{\partial X_2}{\partial x_1} & \frac{\partial X_2}{\partial x_2} & \frac{\partial X_2}{\partial x_3} \\ \frac{\partial X_3}{\partial x_1} & \frac{\partial X_3}{\partial x_2} & \frac{\partial X_3}{\partial x_3} \end{pmatrix}$$

H_{ij}

- You can also form ***Material Displacement Gradient.***

$$\frac{\partial s_i}{\partial X_j} = \frac{\partial x_i}{\partial X_j} - \delta_{ij} = F_{ij} - \delta_{ij}$$

(Recall $s_i = x_i - X_i$)

As obvious, the material displacement gradient is also a tensor.

- In similar lines, ***spatial displacement gradient tensor.*** can also be formulated as follows:

$$\frac{\partial s_i}{\partial x_j} = \delta_{ij} - \frac{\partial X_i}{\partial x_j} = \delta_{ij} - H_{ij}$$

The Deformation Tensors

To know about deformation, the procedure is to see how much change is there between positions of two particles from their initial configuration (at $t=t_0$) and later configuration (at t=t).

- Consider the figure below where the material coordinates $OX_1X_2X_3$ and spatial coordinates $ox_1x_2x_3$ are merged.

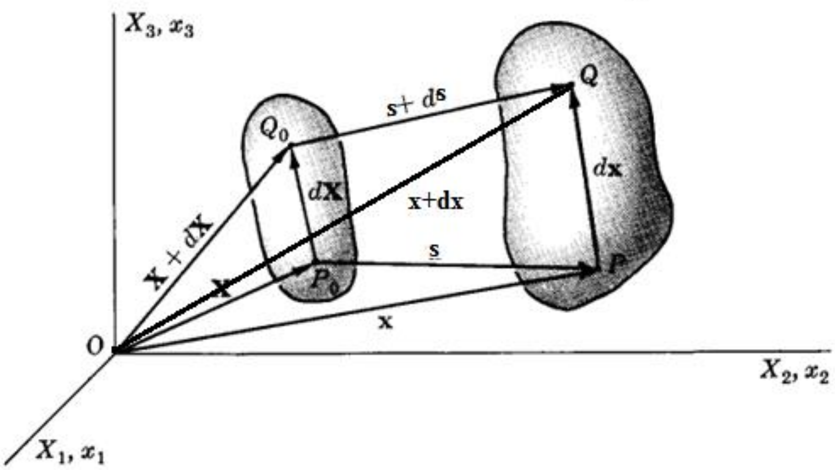

Fig. 2: The deformation tensor representation
(Source: Schaum's outline of theory and problems of continuum mechanics by George Mase)

- There are two neighboring particles that occupy positions P_0 and Q_0 initially at time $t=t_0$.
- The differential elemental element length between two particles is $d\vec{X}$ as per vector algebra.

- After a certain time, at the instant t=t, the continuum has moved as well as deformed. The positions of those particles are given in spatial coordinates $\vec{x}$ and $\vec{x}+d\vec{x}$
- The square of the differential element length between P_0 and Q_0 is:
  $(dX)^2 = d\vec{X}.d\vec{X}$

  In index notation, $(dX)^2 = dX_i.dX_i$

  From $X_i=X_i(x_1,x_2,x_3,t)$, you have seen:

$$\frac{\partial X_i}{\partial x_j} = F_{ij}$$

$$dX_i = \frac{\partial X_i}{\partial x_j} dx_j$$

$$(dX)^2 = \frac{\partial X_i}{\partial x_j} dx_j \frac{\partial X_i}{\partial x_k} dx_k$$

$$= \frac{\partial X_i}{\partial x_j} \frac{\partial X_i}{\partial x_k} dx_j dx_k$$

$$= C_{ij} dx_j dx_k$$

Where C_{ij} → Cauchy's deformation tensor.

- In the deformed configurations, where the particles are at positions P and Q,

$$(dx)^2 = d\vec{x}.d\vec{x}$$

$$(dx)^2 = dx_i dx_i$$

Also from Lagrangian expression, $x_i = x_i(X_1, X_2, X_3, t)$

$$dx_i = \frac{\partial x_i}{\partial X_j} dX_j$$

$$(dx)^2 = dx_i dx_i = \left[\frac{\partial x_k}{\partial X_i} \frac{\partial x_k}{\partial X_j} \right] dX_i dX_j$$

$$(dx)^2 = G_{ij} dX_i dX_j$$

Where G_{ij} → Green's deformation tensor.

- The measure of deformation is evaluated based on the difference $(dx)^2 - (dX)^2$ for the two neighboring particles.

$$(dx)^2 - (dX)^2 = \left[\frac{\partial x_k}{\partial X_i} \frac{\partial x_k}{\partial X_j} \right] dX_i dX_j - dX_i dX_i$$

$$= \left[\frac{\partial x_k}{\partial X_i} \frac{\partial x_k}{\partial X_j} - \delta_{ij} \right] dX_i dX_j$$

$$= 2L_{ij} dX_i dX_j$$

Here L_{ij} → Lagrangian or Green's finite strain tensor

$$L_{ij} = \frac{1}{2} \left[\frac{\partial x_k}{\partial X_i} \frac{\partial x_k}{\partial X_j} - \delta_{ij} \right]$$

- In a similar way, you can form Eulerian strain tensor

  $$E_{ij} = \frac{1}{2}\left[\delta_{ij} - \frac{\partial X_k}{\partial x_i}\frac{\partial X_k}{\partial x_j}\right]$$

- With this brief background information on:
  1. Material coordinates $OX_1X_2X_3$
  2. Spatial coordinates $ox_1x_2x_3$
  3. Material derivative gradient $\rightarrow \frac{\partial x_i}{\partial X_j}$
  4. Spatial derivative gradient $\rightarrow \frac{\partial X_i}{\partial x_j}$
  5. Material displacement gradient $\rightarrow \frac{\partial s_i}{\partial X_j} = \frac{\partial x_i}{\partial X_j} - \delta_{ij}$
  6. Spatial displacement gradient $\rightarrow \frac{\partial s_i}{\partial x_j} = \delta_{ij} - \frac{\partial X_i}{\partial x_j}$
  7. Cauchy's deformation tensor $\rightarrow \frac{\partial X_i}{\partial x_j}\frac{\partial X_i}{\partial x_k}$
  8. Green's deformation tensor $\rightarrow dx_i dx_i = \frac{\partial x_k}{\partial X_i}\frac{\partial x_k}{\partial X_j}$
  9. Lagrangian's finite strain tensor , etc.

- For fluids, we can describe properties in Eulerian or Lagrangian way.
  i.e., For example, the density in the material description will be:
  $\rho = \rho(X_1, X_2, X_3, t)$ i.e. $\rho = \rho(X_i, t)$
  This will be the density of the fluid particle at the position (X_1, X_2, X_3).
- In Eulerian form : $\rho = \rho(X_i(x_1, x_2, x_3, t), t) = \rho(X_i(\mathbf{x}, t), t) = \rho^*(x_i, t)$