Control Volume Approach & Reynolds Transport Theorem

Recall, in the last class we were discussing about a control volume.

- We took the duster, and in solid mechanics, we called this duster as a system.
- You were able to directly apply the principles of conservation of mass, linear momentum, etc. directly on the system to interpret the mechanics.
- For the duster, as its mass is constant and the particles inside are same, the above conservation principle were easy to apply.

- However, in fluid mechanics, it is difficult to analyze a system (or volume) from fluid by considering or tracking the same particle.
- Therefore, in fluids we assumed a definite volume in space that forms the required environment and we can apply mechanics principles on the volume. Such volumes are called control volumes.

- To analyze control volume, we need to convert the mechanics principles that were applicable to a system to the form of control volume.

- If you take a control volume of a liquid, where it is flowing, you can visualize that a fluid system that was initially occupying the control volume will be replaced by another fluid system at the next instant (i.e., the fluid particles are changing).
- To convert the systems analysis conservation concept to a control volume conservation concept, we need to appropriately relate, conceptually as well as mathematically, both of them. **How??**

- The conversion from system analysis to control volume analysis is represented by Reynolds Transport Theorem.

<table>
<thead>
<tr>
<th>System</th>
<th>Control Volume</th>
</tr>
</thead>
<tbody>
<tr>
<td>• Some fluid property of fluid described in space.</td>
<td>• Volume occupying a space and have a shape.</td>
</tr>
<tr>
<td>• Separated from its surroundings by boundaries.</td>
<td>• Volume consists of surfaces called control surfaces.</td>
</tr>
<tr>
<td>• The particles inside the system will be same throughout.</td>
<td>• The fluid particles inside will continuously change.</td>
</tr>
</tbody>
</table>
How will you do Control Volume Analysis?

**Volume and Mass Flow Rate**

Considering an arbitrary volume of liquid in space: It is separated from its surroundings by control surfaces.

- Take a small elemental area $\Delta A$ on the control surface of the volume. The outward normal of the elemental area is $\hat{n}$ as shown in Fig. 1

- Let the velocity vector of fluid passing through the elemental area be $\vec{v}$.
- $\hat{n}$ and $\vec{v}$ may not be collinear.
- The volume of fluid that will sweep through the elemental area $\Delta A$ in an elemental time $\Delta t$ will be:

$$\Delta V = \vec{v} \cdot \Delta t \cdot \Delta A \cos \theta$$

$$\Rightarrow \Delta V = (\vec{v} \cdot \hat{n}) \Delta A \Delta t$$

(The component of velocity vector in the direction of $\hat{n}$ or the component of area vector in the direction of $\vec{v}$)
\[ \frac{\Delta V}{\Delta t} = (\vec{v} \cdot \hat{n}) \Delta A \]

where, \( \frac{\Delta V}{\Delta t} \) = volume flow rate through the elemental area \( \Delta A \). Also, you know, on integrating \( \Delta A \) throughout, you will get the total surface area of the control volume.

Therefore, to get the total volume rate of flow \( Q \) through \( S \), we will first limit the elemental area \( \Delta A \)

\[ \lim_{\Delta t \to 0, \Delta A \to 0} \frac{\Delta V}{\Delta t} = \frac{dV}{dt} \]

\[ Q = \int_S \frac{dV}{dt} = \int_S (\vec{v} \cdot \hat{n}) dA \]

where , \( Q \) = total volume flow rate.

If the fluid concerned in the control volume has a density \( \rho \), then Mass flow rate

\[ m = \int_S \rho (\vec{v} \cdot \hat{n}) dA \]

### Extensive and Intensive Property

For the control volume of the fluid, let \( B \) be any property of the fluid that is related to mass. (e.g., Mass, Momentum, Energy, etc.).

This \( B \) is called the extensive property.

Similarly, we can define another property

\[ \beta = \frac{dB}{dm} \] (i.e., amount of \( B \) per unit mass in any element of the fluid). where \( \beta \) is the intensive property.
To develop Reynolds Transport Theorem

Fig. 2: Nomenclature to derive the Reynolds Transport Theorem
(Source: Dynamics of Fluids in Porous Media by Jacob Bear)

- Let us assume a control volume of a fluid (shown in solid black colour) at an instant “t”.
- As the control volume was chosen at the instant “t”, the fluid particles inside will be unique.
- This is as good as a system (e.g. the duster).
- However, as the fluid is moving, at another instant t+Δt, let the fluid particles that formed the system at time “t” be shifted to a new position and it occupies another location (shown in dotted lines).
- Therefore, at t+Δt, the fluid particles in the control volume is different from that at time “t”.
- At instant “t”, the extensive property:
  \[ B_{CV} = \int_{CV} \beta \rho dU \]
  Where \( \rho \) is the density of the fluid.
The extensive property in the control volume changes due to the following reasons:

1. Time rate of change of B within the control volume

\[
\frac{d(B_{CV})}{dt} = \frac{d}{dt} \int_{CV} \beta \rho dU
\]

2. The outflow of the property B through the surfaces of the control volume

\[
\int_{CS} \beta \rho (\mathbf{\tilde{v}} \cdot \mathbf{n}) dA_{out}
\]

where \( \mathbf{\tilde{v}} \cdot \mathbf{n} \) will be positive.

3. The inflow of the property B through the surfaces of the control volume

\[
\int_{CS} \beta \rho (\mathbf{\tilde{v}} \cdot \mathbf{n}) dA_{in}
\]

where \( \mathbf{\tilde{v}} \cdot \mathbf{n} \) will be negative.

- The inflow and outflow can be marked as net outflow.

- Extensive property in the control volume changes can be summed as:

\[
= \frac{d}{dt} \left[ \int_{CV} \beta \rho dU \right] + \int_{CS} \beta \rho (\mathbf{\tilde{v}} \cdot \mathbf{n}) dA
\]

Note that this representation is Eulerian.

However, as said earlier, the conservation principles can only be directly applied to the system.

For that, let us take \( \Delta t \to 0 \).

Then the control volume and system volume will be the same.

\[
\frac{dB}{dt}_{\text{system}} = \left. \frac{dB}{dt} \right|_{\text{control volume}}
\]

\[
= \frac{d}{dt} \left[ \int_{CV} \beta \rho dU \right] + \int_{CS} \beta \rho (\mathbf{\tilde{v}} \cdot \mathbf{n}) dA
\]

That is, we can relate the time rate of change of property B stored in the system with respect to that of the control volume. The above equation is Reynolds Transport Theorem.