07.02.2017

Lecture 12

Control Volume Approach & Reynolds Transport Theorem

Recall, in the last class we were discussing about a control volume.

- We took the duster, and in solid mechanics, we called this duster as a system.
- You were able to directly apply the principles of conservation of mass, linear momentum, etc. directly on the system to interpret the mechanics.
- For the duster, as its mass is constant and the particles inside are same, the above conservation principle were easy to apply.

- However, in fluid mechanics, it is difficult to analyze a **system** (or volume) from fluid by considering or tracking the same particle.
- Therefore, in fluids we assumed a definite volume in space that forms the required environment and we can apply mechanics principles on the volume. Such volumes are called **control volumes.**

- To analyze control volume, we need to convert the mechanics principles that were applicable to a system to the form of control volume.

- If you take a control volume of a liquid, where it is flowing, you can visualize that a fluid system that was initially occupying the control volume will be replaced by another fluid system at the next instant (i.e., the fluid particles are changing).
- To convert the systems analysis conservation concept to a control volume conservation concept, we need to appropriately relate, conceptually as well as mathematically, both of them. **How??**

  - The conversion from system analysis to control volume analysis is represented by **Reynolds Transport Theorem.**

System	***Control Volume***
• Some fluid property of fluid described in space.	• Volume occupying a space and have a shape.
• Separated from its surroundings by boundaries.	• Volume consists of surfaces called control surfaces.
• The particles inside the system will be same throughout.	• The fluid particles inside will continuously change.

How will you do Control Volume Analysis?

Volume and Mass Flow Rate

Considering an arbitrary volume of liquid in space: It is separated from its surroundings by control surfaces.

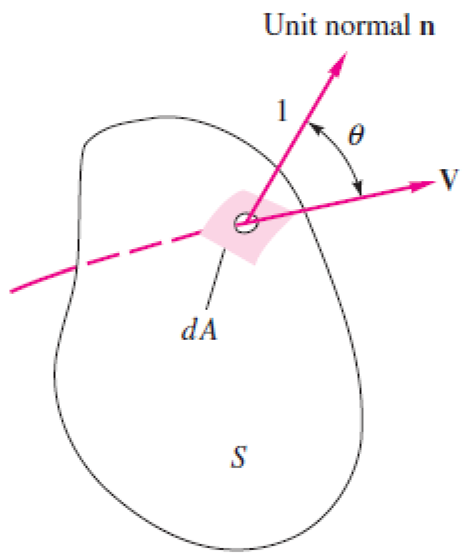

Fig. 1: Elemental area on control surface representation
(Source: Fluid Mechanics by F.M. White)

- Take a small elemental area ΔA on the control surface of the volume. The outward normal of the elemental area is $\hat{n}$ as shown in Fig. 1
    - Let the velocity vector of fluid passing through the elemental area be $\vec{v}$.
    - $\hat{n}$ and $\vec{v}$ may not be collinear.
    - The volume of fluid that will sweep through the elemental area ΔA in an elemental time Δt will be :

$$\Delta V = \vec{v}\ \Delta t\ \Delta A \cos\theta$$

$$\Rightarrow \Delta V = (\vec{v} \cdot \hat{n})\ \Delta A\ \Delta t$$

(The component of velocity vector in the direction of $\hat{n}$ or the component of area vector in the direction of $\vec{v}$)

⇨ $\frac{\Delta V}{\Delta t} = (\vec{v} \,.\, \hat{n})\,\Delta A$

where, $\frac{\Delta V}{\Delta t}$ = volume flow rate through the elemental area ΔA. Also, you know, on integrating ΔA throughout, you will get the total surface area of the control volume.

❖ Therefore, to get the ***total volume rate of flow*** Q through S, we will first limit the elemental area ΔA

⇨ $\lim_{\Delta t \to 0, \Delta A \to 0} \frac{\Delta V}{\Delta t} = \frac{dV}{dt}$

⇨ $Q = \int_S \frac{dV}{dt} = \int_S (\vec{v}.\hat{n})dA$

where , Q = total volume flow rate.

❖ If the fuid concerned in the control volume has a density ρ, then ***Mass flow rate***

$\overset{\circ}{m} = \int_S \rho(\vec{v}.\hat{n})dA$

Extensive and Intensive Property

For the control volume of the fluid, let B be any property of the fluid that is related to mass. (e.g., Mass, Momentum, Energy, etc.).
This B is called the **extensive** property.

Similarly, we can define another property
$\beta = \frac{dB}{dm}$ (i.e., amount of B per unit mass in any element of the fluid).
where β is the intensive property.

To develop Reynolds Transport Theorem

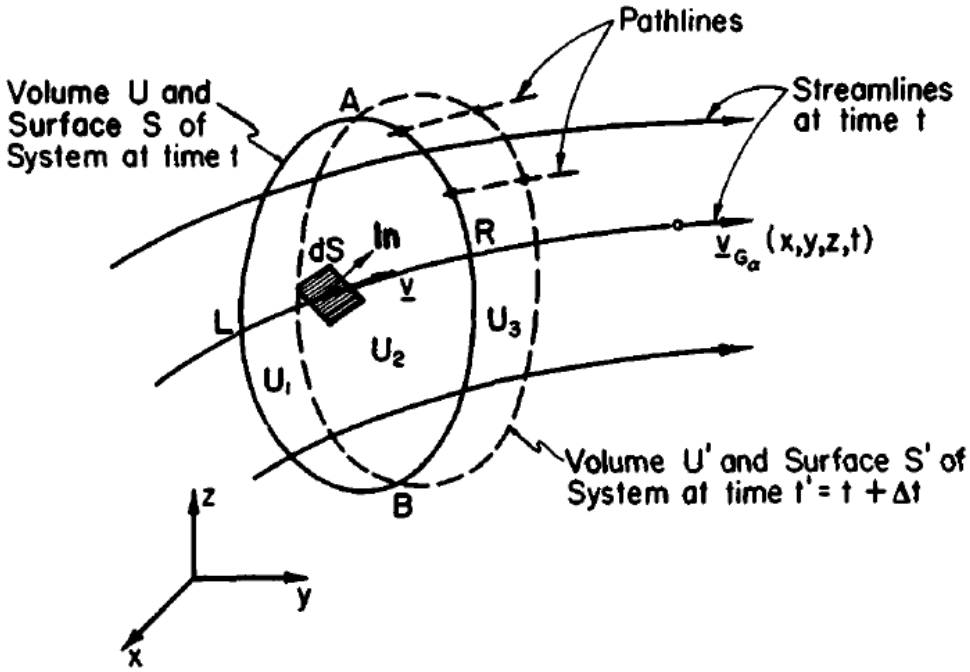

Fig. 2: Nomenclature to derive the Reynolds Transport Theorem
(Source: Dynamics of Fluids in Porous Media by Jacob Bear)

- Let us assume a control volume of a fluid (shown in solid black colour) at an instant "t".
- As the control volume was chosen at the instant "t", the fluid particles inside will be unique.
- This is as good as a system (e.g. the duster).
- However, as the fluid is moving, at another instant t+Δt, let the fluid particles that formed the system at time "t" be shifted to a new position and it occupies another location (shown in dotted lines).
- Therefore, at t+Δt, the fluid particles in the control volume is different from that at time "t".
- At instant "t", the extensive property:

  $$B_{CV} = \int_{CV} \beta \rho dU$$

  Where ρ is the density of the fluid.

- The extensive property in the control volume changes due to the following reasons:
  1. Time rate of change of B within the control volume

     $$\frac{d(B_{CV})}{dt} = \frac{d}{dt}[\int_{CV} \beta \rho dU]$$

  2. The outflow of the property B through the surfaces of the control volume $\int_{CS} \beta \rho (\vec{v}.\hat{n}) dA_{out}$

     where $\vec{v}.\hat{n}$ will be positive.
  3. The inflow of the property B through the surfaces of the control volume $\int_{CS} \beta \rho (\vec{v}.\hat{n}) dA_{in}$

     where $\vec{v}.\hat{n}$ will be negative.

- The inflow and outflow can be marked as net outflow.

- Extensive property in the control volume changes can be summated as:

$$= \frac{d}{dt}[\int_{CV} \beta \rho dU] + \int_{CS} \beta \rho (\vec{v}.\hat{n})\, dA$$

Note that this representation is **Eulerian**.

However, as said earlier, the conservation principles can only be directly applied to the system.

For that, let us take Δt → 0.

Then the control volume and system volume will be the same.

$$\left.\frac{dB}{dt}\right|_{\text{system}} = \left.\frac{dB}{dt}\right|_{\text{control volume}}$$

$$\left.\frac{dB}{dt}\right|_{\text{system}} = \frac{d}{dt}[\int_{CV} \beta \rho dU] + \int_{CS} \beta \rho (\vec{v}.\hat{n})\, dA$$

That is, we can relate the time rate of change of property B stored in the system with respect to that of the control volume. The above equation is ***Reynolds Transport Theorem***.