25/03/2017

LECTURE – 29

Navier – Stokes Equations

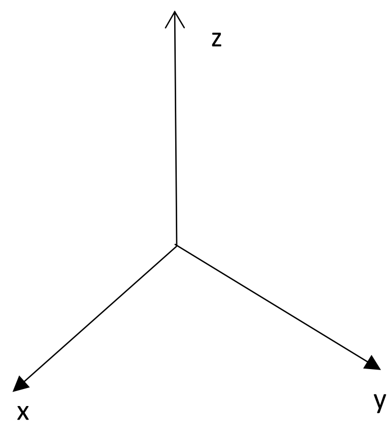
We were discussing the momentum equations in expanding form:

$$\rho g_x - \frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = \rho[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}]$$

$$\rho g_y - \frac{\partial p}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} = \rho[\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}]$$

$$\rho g_z - \frac{\partial p}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} = \rho[\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}]$$

- You have also seen in <u>index notation</u>, the viscous stress is:
  $\tau_{ij} = 2\mu S_{ij} + \lambda S_{mm}\delta_{ij}$



Where, S_{ij} is strain rate tensor

μ, λ is the constants to related viscous stress with strain rates.

- In <u>expanded form</u>:

$$\tau_{xx} = 2\mu S_{xx} + \lambda[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}]\delta_{mm} \qquad \bar{\bar{\delta}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= 2\mu\frac{\partial u}{\partial x} + \lambda(\nabla.\vec{v}) \qquad S_{xx} = \frac{1}{2}[\frac{\partial u}{\partial x} + \frac{\partial u}{\partial x}]$$

For incompressible liquids,

$(\nabla.\vec{v}\ ) = 0$

So, $\tau_{xx} = 2\mu\frac{\partial u}{\partial x}$

Again, $\tau_{yy} = 2\mu\frac{\partial v}{\partial y}$ & $\tau_{zz} = 2\mu\frac{\partial w}{\partial z}$

For the same incompressible fluid,

$\tau_{yx} = \tau_{xy} = \mu\, [\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}]$

$\tau_{zx} = \tau_{xz} = \mu\, [\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}]$

$\tau_{yz} = \tau_{zy} = \mu\, [\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}]$

Therefore, in the expanded form, the set of momentum equations (1) will be for an incompressible fluid:

$$\rho g_x - \frac{\partial p}{\partial x} + \frac{\partial}{\partial x}(2\mu\frac{\partial u}{\partial x}) + \frac{\partial}{\partial y}(\mu(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}) + \frac{\partial}{\partial z}(\mu(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}) = \rho[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}]$$

i.e. $$\rho g_x - \frac{\partial p}{\partial x} + 2\mu\frac{\partial^2 u}{\partial x^2} + \mu\frac{\partial^2 u}{\partial y^2} + \mu\frac{\partial^2 u}{\partial z^2} + \mu\frac{\partial^2 v}{\partial x \partial y} + \mu\frac{\partial^2 w}{\partial x \partial z} = \rho[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}]$$

i.e. $$\rho g_x - \frac{\partial p}{\partial x} + \mu[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}] + \mu\frac{\partial}{\partial x}[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}] = \rho[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}]$$

(As, from continuity equation, $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} = 0$)

i.e. $$\rho g_x - \frac{\partial p}{\partial x} + \mu[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}] = \rho[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}]$$

Similarly,

i.e. $$\rho g_y - \frac{\partial p}{\partial y} + \mu[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}] = \rho[\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}]$$

and

$$\rho g_z - \frac{\partial p}{\partial z} + \mu[\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}] = \rho[\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}]$$

The set of momentum partial differential equation given in (2) is called the famous **Navier Stokes Equation.**

For flow of <u>incompressible Newtonian Fluids</u>

⇨ The Navier-Stokes equations are second-order non-linear partial differential equations.

- The independent variables are x,y,z, and t.
- The dependent variables are p, u,v and w and they are unknowns in the domain.
- There are three momentum equations and four unknowns (p,u,v,w). Hence you have to use the continuity equation for incompressible flow i.e., $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} = 0$ as the fourth equation to simultaneously solve for p,u,v, and w.
- The many famous CFD softwares that use Navier-Stokes equations to solve the fluid flow in any given domain.
- As obvious, the set of four partial differential equations, to be solved, you need to provide appropriate initial and boundary conditions.

Example: (Adopted from FM White's Fluid Mechanics)

For a fluid , it was observed that the velocity components are :

$u = a(x^2-y^2)$, $v = -2axy$, $w=0$. Find the pressure distribution for the given flow?

Ans. Given: $u = a(x^2-y^2)$

$v = -2axy$

$w=0$

Also, $\vec{g} = 0\hat{i} + 0\hat{j} - g\hat{k}$

$$\frac{\partial u}{\partial x} = 2ax, \frac{\partial u}{\partial y} = -2ay$$

Therefore, $\frac{\partial v}{\partial x} = -2ay, \frac{\partial v}{\partial y} = -2ax$

$$\frac{\partial u}{\partial t} = 0, \frac{\partial v}{\partial t} = 0, \frac{\partial w}{\partial t} = 0$$

As the velocity variables are not changing with respect to time, it can be considered that the flow is steady.

- Our objective is to find p(x,y,z,t).
  Since the flow is steady, so we will find p(x,y,z).
  i.e., in the Navier-Stokes equations:

$$\rho g_x - \frac{\partial p}{\partial x} + \mu[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}] = \rho[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}]$$

i.e. $\rho \times 0 - \frac{\partial p}{\partial x} + \mu[2a + (-2a) + 0] = \rho[0 + a(x^2 - y^2)2ax - 2ay(-2ay) + 0]$

i.e. $-\frac{\partial p}{\partial x} = \rho[2a^2 x(x^2 - y^2) + 4a^2 xy^2]$

$= \rho[2a^2 x^3 + 2a^2 xy^2]$

$= 2a^2 \rho x(x^2 + y^2)$

i.e. $\frac{\partial p}{\partial x} = -2a^2 \rho x(x^2 + y^2)$

In y-direction:

$$0 - \frac{\partial p}{\partial y} + \mu[0] = \rho[\frac{\partial v}{\partial t} + \mathrm{a}(\mathrm{x}^2 - \mathrm{y}^2)(-2\,\mathrm{ay}) + (-2\,\mathrm{axy})(-2\,\mathrm{ax})]$$

i.e. $-\frac{\partial p}{\partial y} = \rho[-2a^2x^2\,\mathrm{y} + 2\,\mathrm{a}^2\,y^3 + 4a^2 yx^2]$

i.e. $\frac{\partial p}{\partial y} = -2a^2 \rho y(x^2 + y^2)$

In z-direction:

$$-\rho g - \frac{\partial p}{\partial z} + \mu[0] = \rho[0]$$

i.e. $\frac{\partial p}{\partial z} = -\rho g$

The vertical pressure gradient is hydrostatic.

$$p = \int \frac{\partial p}{\partial x} dx \Big|_{x,y}$$

- The solution $= \int -2\rho a^2 x(x^2 + y^2)dx$

$$= -2\rho a^2[\frac{x^4}{4} + \frac{x^2 y^2}{2}] + f_1(y,z)$$

Again, $\frac{\partial p}{\partial y} = -2\rho a^2 x^2 y + \frac{\partial f_1}{\partial y}$

$$= -2\rho a^2 y(x^2 + y^2)$$

$$\frac{\partial f_1}{\partial y} = -2\rho a^2 y^3$$

or

$$f_1 = \int \frac{\partial f_1}{\partial y} dy \bigg|_z = -2a^2 \rho \frac{y^4}{4} + f_2(\mathrm{z})$$

Comparing,

$$\therefore p = -2\rho a^2 [\frac{x^4}{4} + \frac{x^2 y^2}{2}] - 2a^2 \rho \frac{y^4}{4} + f_2(\mathrm{z})$$

$$\frac{\partial p}{\partial \mathrm{z}} = 0 + \frac{\partial f_2}{\partial \mathrm{z}} = -\rho g$$

$$f_2(\mathrm{z}) = -\rho g\mathrm{z} + c$$

where, c is a constant.

$$p(\mathrm{x}, \mathrm{y}, \mathrm{z}) = -2\rho \, \mathrm{a}^2 [\frac{x^4}{4} + \frac{y^4}{4} + \frac{x^2 y^2}{2}] - \rho g\mathrm{z} + c$$