LECTURE 9

01 - FEB - 2016

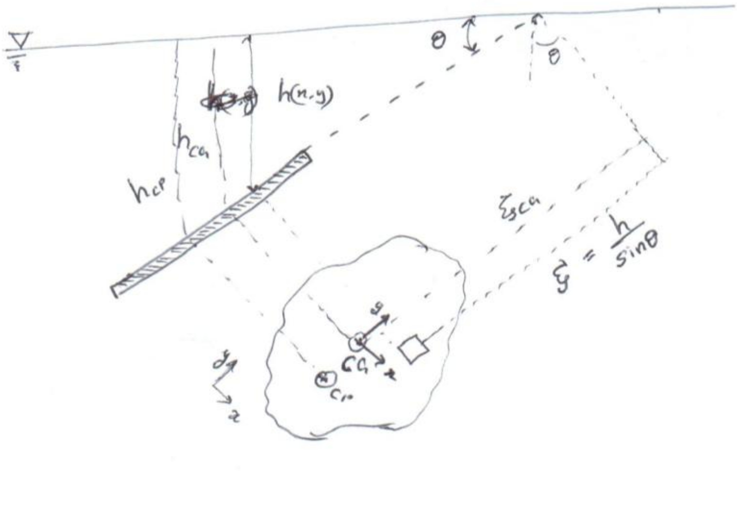
HYDROSTATIC FORCES ON PLANE SURFACES

We introduced to you, in the last class, for water retaining structures like water tanks, reservoirs, dams, etc. the hydrostatic forces also need to be taken into account while designing.

⇒ For a fluid (or liquid), whose density changes are neglected, we have seen that pressure increases with respect to depth.

⇒ There will be a force due to this pressure that acts on the walls, or cielings, or floors, of the retaining structures.

⇒ Consider an arbitrary plane completely submerged in a fluid. The plane is inclined at an angle θ with the free surface of the fluid.



Let 'h' be depth from free surface to any arbitrary ~~point~~ element area dA of the plane.

Pressure at $h(x,y)$ will be $p = p_a + \rho g h$

As shown in figure, the plane area is now ~~oriented~~ described in x-y directions (Please note that this is not the original x, y coordinates of our standard three dimensional co-ordinate system.).

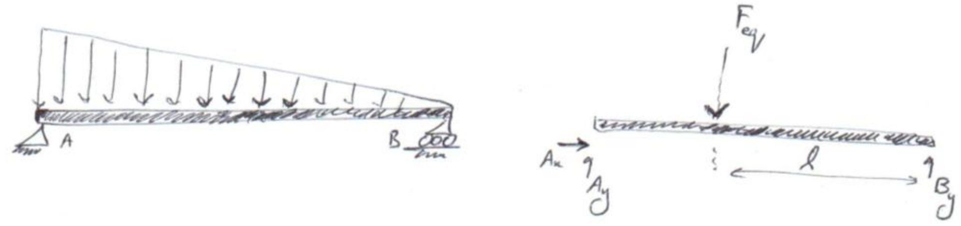
⇒ We can introduce a dummy variable ξ that shows the inclined distance to the arbitrary element from the free surface.

⇒ The total hydrostatic forces on <u>one side</u> of the plate is now given by:

$$F = \int_A p \, dA \qquad ; \text{ where } A \rightarrow \text{Total area of the plane}$$

⇒ Recall, in mechanics, you might have already seen cases having continuously varying loads applied on beams. These continuously varying loads can be replaced appropriately by an equivalent point load such that the moment at any joint of concern should be the same.

i.e.



→ The point of application of equivalent load should be located at a distance 'l' from B, such that the moment at B due to external load should not change.

→ If you look into hydrostatic force on a plane, it is also a case of continuously varying load on the plane.

$$F = \int p \, dA = \int (p_a + \rho g h) \, dA$$

$$= p_a A + \rho g \int h \, dA \quad ; \quad p_a \rightarrow \text{atmospheric pressure}$$

You can see, $h = \xi \sin\theta$.

⇒ Also from the theory of first moment of inertia,

$$\frac{1}{A} \int \xi \, dA = \xi_{CG}$$

, where ξ_{CG} is centroidal slant distance.

(distance to centroid).

→ As the plate is inclined at angle θ, $\sin\theta$ will be a constant.

$$\therefore \quad F = p_a A + \rho g \int \xi \sin\theta \, dA$$

$$= p_a A + \rho g \sin\theta \int \xi \, dA$$

ie. $$F = p_a A + \rho g \sin\theta \, \xi_{CG} A$$

Now $$\xi_{CG} \sin\theta = h_{CG}$$

the depth straight down from free surface to the plate centroid.

$$\therefore \quad F = p_a A + \rho g h_{CG} A$$

$$= \left(p_a + \rho g h_{CG}\right) A$$

$$F = p_{CG} A$$

i.e. Hydrostatic Force on one side of the plane will be

= Pressure at plate centroid * Plane Area.

However, as you have seen for the beam case, when you apply continuously increasing load, it can be replaced by equivalent ~~load~~ point load at ~~a~~ a certain distance from one end.

⇒ Similarly, the hydrostatic force on ~~the~~ one side of the plane can be replaced by the equivalent point load $\boxed{F = p_{CG} A}$. But this point load cannot be applied ~~to~~ ~~at~~ at the area centroid.

It will be applied through the point CP called center of pressure.

(This is because you need to see that the moment due to continuously varying hydrostatic load and equivalent point load at any end should be equal).

$\Rightarrow$ To compute the distance from centroid to the center of pressure, as per the plane co-ordinates x, y can be given as

x_{cp} , y_{cp} .

Let the (x,y) origin be at the centroid $(0,0)$.

$\therefore$ Taking moment about centroid,

$$F\, y_{cp} = \int y\, p\, dA = \int y\,(p_a + \rho g\, \xi \sin\theta)\, dA$$

$$= \rho g \sin\theta \int y\, \xi\, dA + \int y\, p_a\, dA$$

p_a is a constant. $\therefore \int y\, dA = 0$

because you are taking moment with respect to centroidal axes.

$$\therefore\ F\, y_{cp} = \rho g \sin\theta \int y\, \xi\, dA$$

Now, you know $\xi = \xi_{CG} - y$

$$\therefore\ F\, y_{cp} = \rho g \sin\theta \left[\xi_{CG} \int y\, dA - \int y^2\, dA \right]$$

As $\int y \, dA = 0$ (explained earlier)

We have

$$F y_{cp} = -\rho g \sin\theta \int y^2 dA$$

$$= -\rho g \sin\theta \, I_{xx}$$

where I_{xx} is the second moment of inertia of the plane area about its centroidal x-axis.

$$\therefore \quad \boxed{y_{cp} = -\rho g \sin\theta \frac{I_{xx}}{p_{CG} A}}$$

→ The negative sign here indicates that center of pressure usually lies below the centroid of the area.

⇒ Similarly to determine x-coordinate of the center of pressure.

$$F x_{cp} = \int x p \, dA = \int x \left[p_a + \rho g (\xi_{CG} - y) \sin\theta \right] dA$$

Again note $\int x \, dA = 0$, $\int y \, dA = 0$

$$\therefore F x_{cp} = -\rho g \sin\theta \int x y \, dA$$

$$= -\rho g \sin\theta \, I_{xy}$$

where I_{xy} → product of inertia of the plane.

$$\therefore \quad x_{cp} = \frac{-\rho g \sin\theta \; I_{xy}}{p_{CG} \, A}$$

⇒ You can recall that the hydrostatic pressure

$$p_{CG} = p_a + \rho g \, h_{CG}$$

⇒ If we are using gage pressure formulas, then

$p_a = 0$,

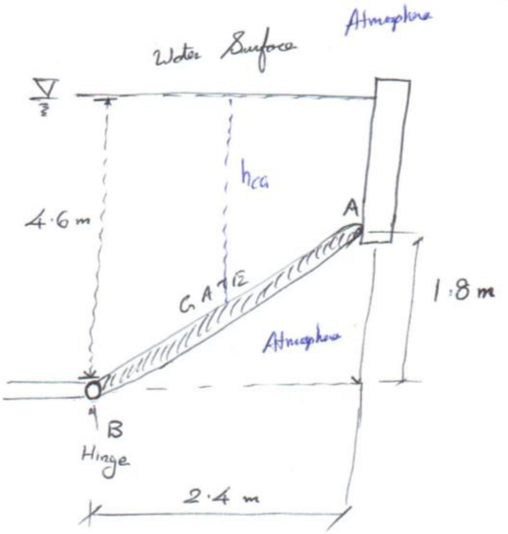
You can have $\quad y_{cp} = -\dfrac{I_{xx} \sin\theta}{h_{CG} \, A}$

and $\quad x_{cp} = -\dfrac{I_{xy} \sin\theta}{h_{CG} \, A}$

Example: (As adopted from FM WHITE's "Fluid Mechanics")

A sluice gate is used to restrict water flow from a tank. The sluice gate is a plane at an inclined angle to the horizontal. The density of water in the tank is 1025 kg/m³. The width of gate is 1.5 m.

a) Compute force on gate due to seawater pressure
b) Horizontal force exerted by wall at point A.
c) Reaction force at hinge B.



→ As can be seen from figure.

→ The length of gate AB

$= \sqrt{2.4^2 + 1.8^2}$

$= \underline{\underline{3.00 \text{ m}}}$

→ ∴ Area of the plate

$= 3.00 * 1.5$

$= 4.50 \text{ m}^2$

⇒ Hydrostatic force on gate $= p_{CG} * A$

→ $h_{CG} = 4.6 - 0.9 = 3.7$ m

$\therefore P_{CG} = \cancel{3.7 *} \ \rho g h_{CG}$ (gage pressure)

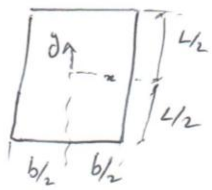
$= 1025 * 9.81 * 3.7$

$\approx 37204 \text{ N/m}^2$

$\therefore F_p = 37204 * 4.5 = \underline{167418 \text{ N}} = \underline{167.42 \text{ kN}}$

⇒ The center of pressure for this hydrostatic force will be:

$$y_{cp} = \frac{-I_{xx} \sin\theta}{h_{CG} A}, \qquad x_{cp} = \frac{-I_{xy} \sin\theta}{h_{CG} A}$$



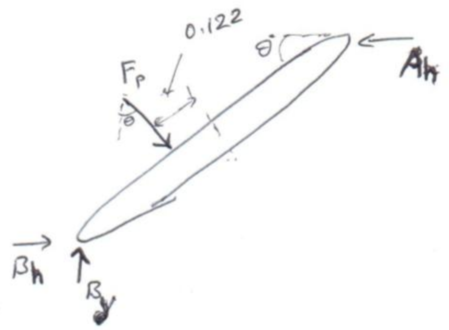
$\therefore \quad I_{xx} = \dfrac{bL^3}{12}$

Here $I_{xx} = \dfrac{1.5 * 3^3}{12} = 3.375 \text{ m}^4$

$I_{xy} = 0 = \int xy \, dA$, $\quad \sin\theta = \dfrac{1.8}{3.0}$

$$\therefore y_{cp} = \frac{-3.375 * 1.8/3.0}{3.7 * 4.5} = -0.122 \text{ m} \ ; \quad x_{cp} = 0.0$$

⇒ Considering free body diagram of gate



$\Sigma F_h = 0$

$\Sigma F_v = 0$

$\Sigma M_B = 0$

i.e. $-A_h + F_p \sin\theta + B_h = 0$

$B_v - F_p \cos\theta = 0$

$B_v - 167.42 * \frac{2.4}{3.0} = 0$

$B_v = 133.94$ kN

Using anti-clockwise moment as +ive.

$\Sigma M_B = 0$; i.e. $A_h * 1.8 - F_p * (1.5 - 0.122) = 0$

i.e. $A_h * 1.8 = 167.42 * 1.28$

$= 214.30$ kN m

$\therefore A_h \approx 119.10$ kN

$B_h = A_h - F_p \sin\theta$

$= 119.10 - 167.42 * \frac{1.8}{3.0}$

$= 18.65$ kN

Reaction at B → $\sqrt{B_h^2 + B_v^2} \approx 135.2$ kN