LECTURE 38

18 - APRIL - 2016

NON - DIMENSIONALISATION OF BASIC EQUATIONS

In the last lecture, we have seen the Buckingham π method to non-dimensionalise the relations or find non-dimensional parameters.

Now recall the basic governing equations of fluid flow:

Consider the liquid to be incompressible.

We had:

$$\vec{\nabla} \cdot \vec{V} = 0 \longrightarrow (1)$$

$$\rho\left[\frac{\partial \vec{V}}{\partial t} + u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}\right] = \rho\vec{g} - \vec{\nabla}p + \mu\nabla^2\vec{V} \longrightarrow (2)$$

The typical boundary conditions were:

* Fixed solid surface : $\vec{V} = 0$
* Inlet or outlet flow: Known $\vec{V}$, p
* If free surface for fluids having negligible effects of surface tension

Free surface $z = \eta$; $w = \dfrac{d\eta}{dt}$

$$p = p_{atm}$$

Now, in these equations, if we go through the dimensions, the three basic dimensions are M, L, T.

→ For incompressible liquid, density ρ is constant

∴ The variables p, $\vec{V}$, x, y, z, t can be non-dimensionalised appropriately.

Let the reference velocity be U and reference length L
(This may be e.g: inlet or upstream velocity U,
L may be diameter of immersed body or it may be characteristic length of aerofoil, etc.).

Define the Non-dimensional variables:

$$\vec{V}^* = \frac{\vec{V}}{U}, \quad \cancel{\nabla^* = \hat{i}\frac{\partial}{\partial x^*}} \quad \text{i.e. } u^* = \frac{u}{U}$$

$$x^* = \frac{x}{L}, \quad y^* = \frac{y}{L}, \quad z^* = \frac{z}{L}, \quad R^* = \frac{R}{L}$$

$$\therefore \nabla^* = \hat{i}\frac{\partial}{\partial x^*} + \hat{j}\frac{\partial}{\partial y^*} + \hat{k}\frac{\partial}{\partial z^*} = L\nabla$$

$$t^* = \frac{tU}{L}, \quad \text{and} \quad p^* = \frac{p + \rho g z}{\rho U^2}$$

So please note that in p^*, we used the piezometric head $p + \rho g z$ in the numerator,

As ρ, U, L are constants, we have:

$$\frac{\partial u}{\partial x} = \frac{\partial(U u^*)}{\partial(L x^*)} = \frac{U}{L}\frac{\partial u^*}{\partial x^*}$$

$$|||^{ly} \quad \frac{\partial v}{\partial y} = \frac{U}{L}\frac{\partial v^*}{\partial y^*} \quad \text{and} \quad \frac{\partial w}{\partial z} = \frac{U}{L}\frac{\partial w^*}{\partial z^*}$$

$$\therefore \text{Continuity eqn.} \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 = \frac{U}{L}\left[\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} + \frac{\partial w^*}{\partial z^*}\right]$$

$$\text{i.e.} \quad \nabla^* \cdot \vec{V}^* = 0 \quad \rightarrow (1)$$

Similarly for momentum equation ~~[illegible]~~

Note: $$\frac{\partial \vec{V}}{\partial t} = \frac{U^2}{L}\frac{\partial \vec{V}^*}{\partial t^*}$$

$$u = u^* U; \quad v = v^* U; \quad w = w^* U$$

∴ The LHS of Navier-Stokes equation becomes.

$$\rho\left[\frac{U^2}{L}\left(\frac{\partial \vec{V}^*}{\partial t^*}\right) + (u^* U)\frac{U}{L}\frac{\partial \vec{V}^*}{\partial x^*} + (v^* U)\frac{U}{L}\frac{\partial \vec{V}^*}{\partial y^*} + (w^* U)\frac{U}{L}\frac{\partial \vec{V}^*}{\partial z^*}\right]$$

$$= \frac{\rho U^2}{L}\left[\frac{\partial \vec{V}^*}{\partial t^*} + (\vec{V}^* \cdot \nabla^*)\vec{V}^*\right]$$

$$= \frac{\rho U^2}{L}\left[\frac{d\vec{V}^*}{dt^*}\right]$$

Similarly you have to non-dimensionalise RHS of Navier-Stokes eqn.

You will get

$$-\rho\frac{U^2}{L}\nabla^* p^* + \frac{\mu}{\rho U L}\left(\frac{\rho U^2}{L}\right)\nabla^{*2}(\vec{V}^*)$$

or $$\frac{d\vec{V}^*}{dt^*} = -\nabla^* p^* + \frac{\mu}{\rho U L}\nabla^{*2}\vec{V}^* \rightarrow (4)$$

The boundary conditions become.

$\vec{V}^* = 0$ for fixed solid surface

For inlet or outlet known values, $\vec{V}^*$, p^*

For free surface, $z^* = \eta^*$; $w^* = \frac{d\eta^*}{dt}$

$$p^* = \frac{P_{atm}}{\rho U^2} + \frac{gL}{U^2} z^*$$; etc.

You can see dimensionless parameters in the above equations:

e.g.: Reynolds number, $Re = \frac{\rho U L}{\mu}$

is seen in Navier-Stokes equations.

From the boundary condition:

The free surface pressure condition has parameters

Euler Number, $Eu = \frac{p_{atm}}{\rho U^2}$

Froude Number, $Fr = \frac{U^2}{gL}$

In Fluid Mechanics, similarly you can come up with several non-dimensional parameters

(i) Reynolds Number, $Re = \frac{\rho U L}{\mu}$

(ii) Mach Number, $Ma = \frac{U}{a}$, a = speed of sound

(iii) Froude Number, $Fr = \frac{U^2}{gL}$

(iv) Weber Number, $We = \frac{\rho U^2 L}{\gamma}$, $\gamma \rightarrow$ surface tension

(v) Cavitation Number, $Ca = \frac{p - p_v}{\rho U^2}$, etc.

Modeling

We have used dimensional homogeneity and π- theorem using power products to relate the concepts.

⇒ Selection of important variables require experience and good judgement.

e.g: Based on the requirement, whether the engineer can

* Neglect effects of viscous flows
* Whether the temperature effects can be neglected
* suggest whether surface tension is important ?, etc.

⇒ Once we select the variables, we need to perform dimensional analysis.

We experimenters will try to seek Similarity between model tested and the prototype designed.

With sufficient testing, the model data suggest dimensionless function between variables:

i.e. $\Pi_1 = f(\Pi_2, \Pi_3, \Pi_4, \ldots, \Pi_k)$

⇒ Now you can model in such a way that

"Flow conditions for a Model test are completely similar if all relevant dimensionless parameters ~~have the same~~ ~~correspon~~ $\Pi_2, \Pi_3, \Pi_4, \ldots, \Pi_k$ have the same corresponding values for the model and the prototype."

(From FM White's Fluid Mechanics)

i.e. If you have a model for a prototype, the model will be completely similar to the prototype if $\Pi_{2m} = \Pi_{2p}$; $\Pi_{3m} = \Pi_{3p}$; ....

Then you will see that $\Pi_{1m} = \Pi_{1p}$.

⇒ Most of the times, to have complete similarity is quite difficult.

So there are similarities like

→ geometric similarity
→ kinematic similarity
→ dynamic similarity
→ thermal similarity.

Geometric Similarity

"A model and a prototype are geometrically similar if and only if all body dimensions in all three coordinates have same linear scale ratio". (From FM White's Fluid Mechanics)

i.e. $\dfrac{X_m}{X_p} = \dfrac{Y_m}{Y_p} = \dfrac{Z_m}{Z_p}$

All angles will be same in model as well as prototype.

Prototype   Model