LECTURE 37

13 - APRIL - 2016

BUCKHINGHAM PI THEORY

→ Yesterday, we started discussing about Buckhingham Pi Theory.

→ This is a method to form non-dimensional expression for various physical process that satisfy principle of dimensional homogeneity (PDH).

We will describe the steps involved in Buckhingham Pi Theory through our example :

$$F = f(\rho, L, U, \mu) \longrightarrow (1)$$

(i) List and count the n-variables in the problem.

Here the variables are F, ρ, L, U, μ

i.e. $n = 5$

(ii) List the basic dimensions of each variable

F	L	U	ρ	μ
$[MLT^{-2}]$	$[L]$	$[LT^{-1}]$	$[ML^{-3}]$	$[ML^{-1}T^{-1}]$

(Overall, you can see the basic dimensions are 3) MLT

(iii) Find 'j', such that they are variables that do not form a Π - product among themselves.

The basic dimension is 3. Therefore, you may also see three variables (out of 5) that may be independent or combination of them will not form a Π - product.

For this example, we can list such variables

The variable L is independent and have dimension of $[L]$

The variable U is independent as it is the one having dimension $[T]$

The variable ρ is independent as it contain $[M]$.

The remaining two $F \rightarrow$ can be expressed by above three

as well as $\mu \rightarrow$ can be expressed by above three.

$\therefore$ Therefore $j = 3$, here.

$k = n - j = 5 - 3 = 2$.

There will be two Π's

(iv) The scaling parameters that do not form Π- product

Here it is $j = 3$ and they are L, U, ρ

(v) To form Π's by adding one remaining variable to the j-repeating variables to form a power product.

i.e. If $v_1 = f(v_2, v_3, v_4, v_5)$

and if v_2, v_3, v_4 is independent

The number of Π's will be $= 5 - 3 = 2$

$$\Pi_1 = (v_2)^a (v_3)^b (v_4)^c (v_1) = M^0 L^0 T^0$$

$$\Pi_2 = (v_2)^a (v_3)^b (v_4)^c v_5 = M^0 L^0 T^0$$

Similarly, here it will be:

$$\Pi_1 = L^a U^b \rho^c F \rightarrow M^0 L^0 T^0$$

i.e.

$$[L]^a [LT^{-1}]^b [ML^{-3}]^c [MLT^{-2}] = M^0 L^0 T^0$$

or

$$a + b - 3c + 1 = 0$$
$$-b - 2 = 0$$
$$c + 1 = 0$$

$\therefore c = -1$, $b = -2$, $\therefore a = -2$

$$\Pi_1 = L^{-2} U^{-2} \rho^{-1} F = \frac{F}{\rho U^2 L^2}$$ called C_F earlier.

$$\Pi_2 = L^a U^b \rho^c \mu$$

$$[L]^a [LT^{-1}]^b [ML^{-3}]^c [ML^{-1}T^{-1}] = M^0 L^0 T^0$$

or

$$a + b - 3c - 1 = 0$$
$$-b - 1 = 0 \; ; \quad b = -1$$
$$c + 1 = 0 \; ; \quad c = -1$$

$$\therefore a = 1 - 3 + 1 = -1$$

$$\therefore \Pi_2 = L^{-1} U^{-1} \rho^{-1} \mu = \frac{\mu}{\rho U L} = \frac{1}{Re}$$

(vi) The final dimensionless function will be of the form:

$$\frac{F}{\rho U^2 L^2} = \text{some function of } \frac{1}{Re} = g(Re)$$

Example As adopted from (F-M White's Fluid Mechanics)

At low velocities (laminar flow), the volume flow Q through a small-bore tube is a function only of the tube radius R, the fluid viscosity μ and the pressure drop per unit tube length $\frac{dp}{dx}$. Using Pi-theorem, find an appropriate dimensionless relationship

Answer:

As stated in the problem question, we can now express the volume flow Q as:

$$Q = f\left(R, \mu, \frac{dp}{dx}\right)$$

(i) The number of variables $\rightarrow$ $Q, R, \mu, \frac{dp}{dx}$, 4

$$n = 4.$$

(ii) Using MLT system, the number of basic dimensions

Q	R	μ	dp/dx
$[L^3 T^{-1}]$	$[L]$,	$[ML^{-1}T^{-1}]$	$[ML^{-2}T^{-2}]$

There are three primary dimensions M, L, T

iii) To find repeating variables, you can see R is independent. Should we take dp/dx or μ or both.

Let us check, whether independent non-dimensionless Π-group can be formed within R, μ, and dp/dx.

i.e. $\Pi = R^a \mu^b \left(\frac{dp}{dx}\right)^c$

i.e. $[L]^a \; [ML^{-1}T^{-1}]^b \; [ML^{-2}T^{-2}]^c = M^0 L^0 T^0$

i.e.
$$\left.\begin{aligned} a - b - 2c &= 0 \\ -b - 2c &= 0 \\ b + c &= 0 \end{aligned}\right\}$$
You can see there is conflict between relations of b and c.

That means, we need to incorporate both μ and dp/dx as repeating variables.

$\therefore \; j = 3.$

(iv) The scaling parameters are R, μ, dp/dx

(v) There will be $n - j = 4 - 3 = 1$ pi-group.

$$\Pi_1 = R^a \mu^b \left(\frac{dp}{dx}\right)^c Q$$

i.e. $[L]^a \; [ML^{-1}T^{-1}]^b \; [ML^{-2}T^{-2}]^c \; [L^3 T^{-1}] = M^0 L^0 T^0$

i.e.
$$\begin{aligned} a - b - 2c + 3 &= 0 \quad ; \quad a = -4 \\ -b - 2c - 1 &= 0 \quad \text{gives} \quad c = -1 \\ b + c &= 0 \quad ; \quad b = -c \;, \therefore b = 1 \end{aligned}$$

$$\therefore \; \Pi_1 = R^{-4} \mu \left(\frac{dp}{dx}\right)^{-1} Q$$

$$= \frac{Q\mu}{R^4 \left(dp/dx\right)} = \text{constant}$$

From laminar pipe flow analysis, you see that the constant is equal to $-\frac{\pi}{8} = -0.39276$