LECTURE 34
09 - APRIL - 2015

SOLUTIONS TO NAVIER - STOKES EQUATIONS

Last day, we discussed about strain rate.

e.g. In xy plane,

$$\dot{\varepsilon}_{xy} = \frac{1}{2}\left[\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right]$$

Subsequently, we suggested that in fluids, the strain rate and stresses are related.

i.e. ~~Str~~ For Newtonian fluid, stress is linearly proportional to strain rate

i.e. $\tau_{xy} \propto \dot{\varepsilon}_{xy}$

In higher or advanced fluid mechanics, one can see that viscous stress tensor is linearly related to strain rate tensor.

$$\overline{\overline{\tau}} = \overline{\overline{\overline{K}}}\,\overline{\overline{\dot{\varepsilon}}}$$

where $\overline{\overline{\overline{K}}}$ is a fourth-rank tensor (i.e. it consists of 81 components in three-dimensional coordinates)

→ However, the $\overline{\overline{\tau}}$ is symmetric and assuming fluid as isotropic, you will see

$$\overline{\overline{\tau}} = 2\mu\,\overline{\overline{\dot{\varepsilon}}} + \lambda(\nabla\cdot\vec{v})\,\overline{\overline{\delta}}$$; where $\overline{\overline{\delta}}$ → Kronecker delta (a second-rank tensor).

→ Now for incompressible fluid, $\nabla\cdot\vec{v} = 0$, you have

$$\overline{\overline{\tau}} = 2\mu\,\overline{\overline{\dot{\varepsilon}}}$$

or $$\tau_{xy} = 2\mu \cdot \frac{1}{2}\left[\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right]$$

$$\tau_{xz} = 2\mu \cdot \frac{1}{2}\left[\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right] \quad , \quad etc.....$$

(Therefore, we have now clarified how the stress versus strain rate relations were used in Navier - Stokes equations).

→ You have also seen about potential functions $\phi(x, y, z, t)$ for irrotational flows.

→ Navier - Stokes equations can be used for any type of incompressible fluid flow. For other compressible flows, you can also use the general momentum equations represented earlier.

→ For various problems, one can solve Navier - Stokes equations appropriately.

Couette Flow between fixed and moving plate

Consider fluid flow between two plates (parallel).

Two plates
- → One fixed
- → The other moving at velocity V.

⇒ Plates are very wide. Therefore, flow is axial.

Let us consider steady state situation.

The components of velocity v and w are zero

$u \neq 0$.

Neglecting gravity effects and also assuming there is no pressure gradient

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 = \frac{\partial u}{\partial x} + 0 + 0 = 0$$

From the figure, you can see x and y directions

As $\frac{\partial u}{\partial x} = 0$, i.e. $u = u(y)$ only.

$\therefore$ In the x-component of Navier-Stokes equation:

$$\rho\left[u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right] = -\frac{\partial p}{\partial x} + \rho g_x + \mu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right]$$

i.e. $\rho\left[0+0\right] = 0 + 0 + \mu\left[0 + \frac{d^2 u}{dy^2}\right]$

$$\frac{d^2 u}{dy^2} = 0 \quad \text{or} \quad u = C_1 y + C_2$$

At $y = +h$, $u = V = C_1 h + C_2$

$y = -h$, $u = 0 = -C_1 h + C_2$

$\therefore C_1 = \frac{V}{2h}$ and $C_2 = \frac{V}{2}$

$$\therefore u = \frac{V}{2h}y + \frac{V}{2} = \frac{V}{2}\left(\frac{y}{h} + 1\right)$$

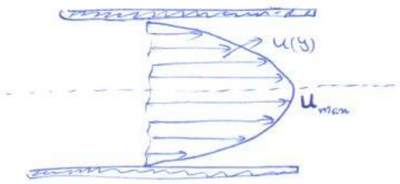
for $-h \le y \le +h$

This is the required solution for Couette flow between two parallel pipes.

Flow due to pressure gradient between two fixed plates

Consider two plates fixed.

Here V = velocity of plate $= 0$.



However, we are now considering the case, where pressure varies in x- direction.

Flow is steady.

As it is two dimensional case, $u = u$, $v = 0$, $w = 0$.

$$\therefore \quad u = u(y) \quad \text{from continuity.}$$

$\therefore$ x- momentum equation becomes.

$$\rho\left[u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right] = -\frac{\partial p}{\partial x} + \mu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right]$$

i.e. $$\rho\left[0 + 0\right] = -\frac{\partial p}{\partial x} + \mu\frac{\partial^2 u}{\partial y^2}$$

i.e. $$\underline{\frac{\partial p}{\partial x} = \mu\frac{\partial^2 u}{\partial y^2}}$$

From y- momentum equation:

$$\rho\left[u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right] = -\frac{\partial p}{\partial y} + \mu\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right]$$

i.e. $$\rho\left[0 + 0\right] = -\frac{\partial p}{\partial y} + \mu\left[0 + 0\right] \quad \text{or} \quad \frac{\partial p}{\partial y} = 0$$

From z- momentum equation

$$\rho\left[0\right] = \rho g_z - \frac{\partial p}{\partial z} + \mu\left[0\right]$$

However as effects of gravity are neglected, $\frac{\partial p}{\partial z} = 0$

$\therefore$ That means $p = p(x)$ only.

~~Hence.~~ ~~$\frac{d^2u}{d}$~~ As $u = u(y)$ only
and $p = p(x)$ only

we can write
$$\mu \frac{d^2u}{dy^2} = \frac{dp}{dx}$$

Again from ~~differentiation~~ calculus principles,

we get $\mu \frac{d^2u}{dy^2} = \frac{dp}{dx} = \text{constant}$

$\Rightarrow$ This constant should be negative, because pressure must decrease in the flow direction so ~~as~~ to overcome the resisting wall shear stress

$\Rightarrow$ The velocity profile $u(y)$ should have a negative curvature everywhere.

$$u = \frac{1}{\mu}\left(\frac{dp}{dx}\right)\frac{y^2}{2} + c_1 y + c_2$$

At $y = \pm h$; ~~$y = -h$,~~ $u = 0$, $\therefore c_1 = 0$

$$c_2 = -\frac{dp}{dx}\frac{h^2}{2\mu}$$

$$\therefore u = -\frac{dp}{dx}\frac{h^2}{2\mu}\left[1 - \frac{y^2}{h^2}\right]$$

This is Poiseuille flow.