LECTURE 33

06 - APRIL - 2016

VORTICITY & POTENTIAL

Yesterday, for the incompressible fluid in steady state, we have derived expressions for angular velocity components. e.g: $\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$

Similarly, we extended the concept for angular velocity in all directions

i.e Angular velocity, $\vec{\omega} = \frac{1}{2}\left(\vec{\nabla} \times \vec{v}\right)$

i.e. $$\vec{\omega} = \frac{1}{2}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ u & v & w \end{vmatrix}$$

⇒ From studies, it is observed that $\vec{\omega}$ may be very small.

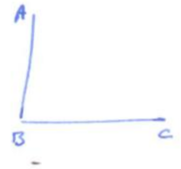
⇒ We can introduce the term VORTICITY, which is nothing but twice of $\vec{\omega}$.

Vorticity, $\vec{\zeta} = 2\vec{\omega} = \text{curl}(\vec{v})$

⇒ For irrotational flows, $\text{curl}(\vec{v}) = 0$

Also you can note that, you can have irrotational flows for incompressible or compressible, steady or unsteady fluids.

⇒ For those two lines, the angular shear strains can also be evaluated.



For line BC,

$$\frac{\frac{\partial v}{\partial x}\Delta x \Delta t \; \cancel{\ }}{\Delta x} = \frac{\partial v}{\partial x}\Delta t = \frac{d\alpha}{dt}\Delta t$$

For line AB,

$$\frac{\frac{\partial u}{\partial y}\Delta y \Delta t \; \cancel{\ }}{\Delta y} = \frac{\partial u}{\partial y}\Delta t = \frac{d\beta}{dt}\Delta t$$

$$\therefore \text{ Average shear strain } = \frac{1}{2}\left[\frac{d\alpha}{dt}\Delta t + \frac{d\beta}{dt}\Delta t\right]$$

$$= \frac{1}{2}\left[\frac{\partial v}{\partial x}\Delta t + \frac{\partial u}{\partial y}\Delta t\right]$$

$$\text{Shear strain rate} \quad \mathring{\varepsilon}_{xy} = \frac{1}{2}\lim_{\Delta t \to 0}\frac{1}{\Delta t}\left(\frac{\partial v}{\partial x}\Delta t + \frac{\partial u}{\partial y}\Delta t\right)$$

$$\text{i.e.} \quad \mathring{\varepsilon}_{xy} = \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$$

⇒ Recall, our explanation of differential equations on linear momentum. There we suggested the relation between ~~shear~~ viscous stress and velocity gradient

$$\text{Shear stress} = \mu \times \text{Shear strain rate}$$

$$\text{e.g.} \quad \tau_{xy} = \mu \, \mathring{\varepsilon}_{xy}$$

$$= \frac{\mu}{2}\left[\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right]$$

Similarly other shear stress components τ_{yz} and τ_{xz} can also be expressed in terms of velocity gradients.

$\Rightarrow$ You know, for irrotational flows $\vec{\nabla} \times \vec{v} = 0$

Using vector calculus principles, we can now write $\vec{\nabla} \times \vec{v} = 0 = \vec{\nabla} \times \nabla \phi$

where ϕ is a scalar function in x and y (for two - dimensional flows).

In three - dimensional flows: $\phi \rightarrow \phi(x, y, z, t)$

where $u = \frac{\partial \phi}{\partial x}$, $v = \frac{\partial \phi}{\partial y}$, $w = \frac{\partial \phi}{\partial z}$

This $\phi(x, y, z, t)$ is called POTENTIAL FUNCTION.

$\rightarrow$ That means that this potential functions exist for irrotational flows.

In civil engineering, there are many fluid flow problems that ~~have~~ are irrotational.

$\rightarrow$ So one can use potential function in a flow domain to solve the flow equations.

Note:-

If your fluid flow is irrotational as well as in two - dimensions, then both ψ and ϕ exist. You can draw streamlines and potential lines everywhere in the flow domain (except at stagnation points where $\vec{v} = 0$)

i.e. $u = \frac{\partial \psi}{\partial y} = \frac{\partial \phi}{\partial x}$

$v = \cancel{\frac{\partial \psi}{\partial y}} - \frac{\partial \psi}{\partial x} = \frac{\partial \phi}{\partial y}$

You have seen for irrotational flows incompressible liquids $\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$

Similarly, $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$

$$d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy$$

$$= u\, dx + v\, dy$$

For a line of constant ϕ,

you have $d\phi = 0 = u\,dx + v\,dy$

Hence $\left(\frac{dy}{dx}\right)_{\phi = \text{constant}} = -\frac{u}{v} = \frac{-1}{\left(\frac{dy}{dx}\right)_{\psi = \text{constant}}}$

From streamline concept,

$\frac{dx}{u} = \frac{dy}{v}$

or $v\,dx - u\,dy = 0$

or $-\frac{\partial \psi}{\partial x} dx - \frac{\partial \psi}{\partial y} dy = 0$

or $d\psi = 0$.

$\left(\frac{dy}{dx}\right)_{\psi = \text{const}} = \frac{v}{u}$

This implies that lines of constant ϕ and constant ψ are mutually orthogonal.

⇒ Recall, when we described about Bernoulli's equation. It was used for frictionless fluid:

→ Along a streamline between two points 1 and 2

$$\int_1^2 \frac{\partial \mathbf{V}}{\partial t} ds + \int_1^2 \frac{dp}{\rho} + \frac{1}{2}\left(V_2^2 - V_1^2\right) + g(z_2 - z_1) = 0$$

→ For steady incompressible flow, you got

$$\frac{p}{\rho} + \frac{V^2}{2} + gz = \text{constant along a streamline}$$

→ This constant can vary from streamline to streamline.

→ If the flow is irrotational $\vec{\zeta} = \vec{\nabla} \times \vec{v} = 0$

Then the equation $\frac{p}{\rho} + \frac{V^2}{2} + gz = \text{constant}$

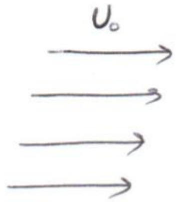
throughout the fluid domain.

→ So Bernoulli's equation is valid for frictionless fluid.

Q: When is a flow irrotational?

<u>Ans:</u> For irrotational flow, you have $\nabla^2 \phi = 0$

e.g. Uniform flow.



⇒ In the previous example problem $u = a(x^2 - y^2)$,

$v = -2axy$, $w = 0$.

We can check, whether the flow is irrotational

$$\nabla \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a(x^2-y^2) & -2axy & 0 \end{vmatrix} = \hat{k}\left(-2ay + 2ay\right) = \underline{\underline{0}}$$

∴ Flow is irrotational. You can form <u>potential function</u>.

⇒ In reality, most of the situation we have viscous flows. That is, no-slip conditions prevail. You may not see irrotational effects. You have to use Navier-Stokes equations to solve fluid flow.