LECTURE 32
05-APRIL-2016

STREAM FUNCTION, VORTICITY, POTENTIAL

Yesterday, we introduced you the concept of stream function for an incompressible, steady two-dimensional fluid flow.

$\psi(x,y) \rightarrow$ Stream function

$$u = \frac{\partial \psi}{\partial y} \quad ; \quad v = -\frac{\partial \psi}{\partial x}$$

Example (as adopted from FM White's Fluid Mechanics)

In a flow field, the velocity components were ~~measured~~ evaluated as $u = a(x^2 - y^2)$; $v = -2axy$; $w = 0$. Check whether you can form a stream function for this flow field. If so, what is the stream function?

Ans Given

$$u = a(x^2 - y^2)$$
$$v = -2axy$$
$$w = 0$$

$\therefore$ The flow is two-dimensional as $w = 0$.

We need to check, whether the incompressible continuity equation is satisfied. i.e. $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$

$$\frac{\partial u}{\partial x} = 2ax, \quad \frac{\partial v}{\partial y} = -2ax$$

$\therefore$ Continuity equation is satisfied.

Hence stream function ψ exist for the given problem.

To formulate ψ :

$$u = \frac{\partial \psi}{\partial y} = a x^2 - a y^2$$

$$\therefore \quad \psi = \int u \, dy + f(x)$$

$$\text{i.e.} \quad \psi = a x^2 y - \frac{a y^3}{3} + f(x)$$

$$\therefore \quad \frac{\partial \psi}{\partial x} = 2axy + f'(x)$$

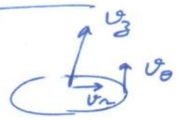
However, $\dfrac{\partial \psi}{\partial x} = -v = 2axy = 2axy + f'(x)$

$$\therefore \quad f'(x) = 0 \quad \text{or} \quad f(x) = \text{constant} = C$$

$$\therefore \quad \psi = a x^2 y - \frac{a y^3}{3} + C$$

Incompressible Plane Flow in Polar Coordinates

Recall in polar co-ordinates you have v_r, v_θ, and v_z as velocity components.



→ In a planar flow, let $v_z = 0$..

Equation of continuity for incompressible fluid will be

$$\frac{1}{r} \frac{\partial}{\partial r}(r v_r) + \frac{1}{r} \frac{\partial}{\partial \theta}(v_\theta) = 0$$

or $\dfrac{\partial}{\partial r}\left(\dfrac{\partial \psi}{\partial \theta}\right) + \dfrac{\partial}{\partial \theta}\left(-\dfrac{\partial \psi}{\partial r}\right) = 0$;

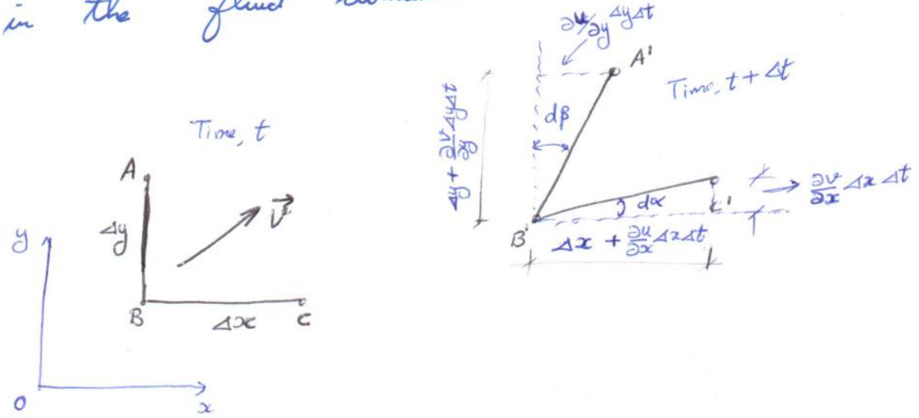
where $v_r = \dfrac{1}{r}\dfrac{\partial \psi}{\partial \theta}$ and $v_\theta = -\dfrac{\partial \psi}{\partial r}$

Here again, lines of constant ψ's will be streamlines.

Also $\quad d\varphi = d\psi$

Vorticity and Irrotationality

We can now talk about angular deformation and angular velocity of fluid.
Let us take the same two-dimensional flow in xy-plane. We will consider two lines perpendicular to each other initially (Line 1 & Line 2) in the fluid domain. Lines AB and BC



→ Lines AB has length Δy initially at time 't'.

→ Line BC has length Δx at time 't'.

→ At time $t + \Delta t$, these lines deform and have the positions as shown A'B' and B'C'

→ Let the angular deformation be considered positive in counter-clockwise direction.

So you ~~can~~ can see at time $t + \Delta t$, you have $+ d\alpha$ and $- d\beta$

→ We can now form differential relations for deformation of a fluid element.

→ The deformation here occurs kinematically because of spatial gradients in the velocity vector $\vec{v}$.

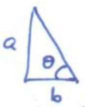
→ You know angular velocity is defined as the rate of change of angular displacement (or deformation)

∴ Angular velocity ω_z about the z- axis will be the average rate of counter clockwise turning of the two lines :

i.e. $$\omega_z = \frac{1}{2} * \left[\frac{d\alpha}{dt} + \left(-\frac{d\beta}{dt}\right)\right]$$

$$= \frac{1}{2}\left(\frac{d\alpha}{dt} - \frac{d\beta}{dt}\right)$$

i.e. From the figure, we tan relation ⟶



$\theta = \tan^{-1}\left(\frac{a}{b}\right)$

$$\frac{d\alpha}{dt} = \underset{\Delta t \to 0}{Lt}\left[\frac{1}{\Delta t} \tan^{-1} \frac{\frac{\partial v}{\partial x}\Delta x \Delta t}{\Delta x + \frac{\partial u}{\partial x}\Delta x \Delta t}\right] = \frac{\partial v}{\partial x}$$

$$\frac{d\beta}{dt} = \underset{\Delta t \to 0}{Lt}\left[\frac{1}{\Delta t} \tan^{-1} \frac{\frac{\partial u}{\partial y}\Delta y \Delta t}{\Delta y + \frac{\partial v}{\partial y}\Delta y \Delta t}\right] = \frac{\partial u}{\partial y}$$

$$\therefore \omega_z = \frac{1}{2}\left[\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right]$$

Similarly in general, you can have $\omega_x = \frac{1}{2}\left[\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right]$

and $\omega_y = \frac{1}{2}\left[\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right]$

Angular velocity vector $\vec{\omega} = \omega_x \hat{i} + \omega_y \hat{j} + \omega_z \hat{k}$

Also $\vec{\omega}$, ~~you~~ you can see now will be:

$$\vec{\omega} = \frac{1}{2}(\vec{\nabla} \times \vec{v}) = \frac{1}{2}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ u & v & w \end{vmatrix}$$

As you see that $\vec{\omega} = \frac{1}{2}(\vec{\nabla} \times \vec{v})$ and from most observations, it is found to be small, scientists came up with another term Vorticity, which is nothing but the twice of $\vec{\omega}$

i.e. Vorticity $\vec{\zeta} = 2\vec{\omega} = \text{curl}(\vec{V})$

$\Rightarrow$ For many flows in civil engineering, you may see that $\text{curl}(\vec{V}) = 0$.

Such flows are called irrotational flows.

(Practically, you can have irrotational flows for incompressible or compressible, steady or unsteady fluids).

$\Rightarrow$ We can also now describe an shear strain rate.

In those two lines, the angular shear strain will be:

$$\left[\frac{\frac{\partial v}{\partial x}\Delta x \Delta t}{\Delta x}\right] = \frac{\partial v}{\partial x}\Delta t \quad \text{or} \quad \frac{d\alpha}{dt}\Delta t$$

$$\frac{\frac{\partial u}{\partial y}\Delta y \Delta t}{\Delta y} = \frac{\partial u}{\partial y}\Delta t = \frac{d\beta}{dt}\Delta t$$

$\therefore$ Average Shear strain $= \frac{1}{2}\left[\frac{\partial v}{\partial x}\Delta t + \frac{\partial u}{\partial y}\Delta t\right]$

Shear strain rate $\dot{\varepsilon}_{xy} = \frac{1}{2}\lim_{\Delta t \to 0}\left(\frac{\partial v}{\partial x}\Delta t + \frac{\partial u}{\partial y}\Delta t\right)\frac{1}{\Delta t}$

$$= \frac{1}{2}\left[\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right]$$