LECTURE 31

04 - APRIL - 2016

STREAM FUNCTION

Last classes, we described about the differential equations that can be used to represent:

* the conservation of mass
* the linear momentum principle
* the energy principle

⇒ We also saw that we need to apply appropriate conditions to solve these differential equations in the respective fluid flow domain.

⇒ For some of the fluid problems, where the flow field varies predominantly in x-, y- directions only and no variations in z- direction, Then the derivatives in the direction of z- can be neglected.

e.g: The continuity equation for incompressible fluid can be given as:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad \rightarrow (1)$$

⇒ To utilise the mathematical property, we can define a scalar function ψ, such that the above continuity equation (1) is satisfied.

That is, 1 can be written as:

$$\frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial \psi}{\partial x}\right) = 0$$

where $\psi(x,y)$ is a function called stream function.

$\Rightarrow$ Now it is clear,

$$u = \frac{\partial \psi}{\partial y} \quad , \quad \text{and} \quad v = -\frac{\partial \psi}{\partial x}$$

$\Rightarrow$ Utilising ψ will help us to reduce the number of unknowns to one (i.e. ψ) and utilise it directly in linear momentum equation to solve for fluid flow in the domain.

i.e. $$\vec{V} = \frac{\partial \psi}{\partial y}\hat{i} - \frac{\partial \psi}{\partial x}\hat{j}$$

Therefore, in the momentum equation

$$\rho\left[\frac{\partial \vec{V}}{\partial t} + u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}\right] = \rho\vec{g} - \vec{\nabla}p + \mu\nabla^2\vec{V}$$

As the flow is in x-, y- directions and also steady, we can write:

$$u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} = \vec{g} - \frac{1}{\rho}\vec{\nabla}p + \frac{\mu}{\rho}\left(\frac{\partial^2 \vec{V}}{\partial x^2} + \frac{\partial^2 \vec{V}}{\partial y^2}\right) \rightarrow 2$$

If we take curl of equation 2, we get:

i.e. $$\vec{\nabla} \times [\quad] = \vec{\nabla} \times [\quad]$$

$$\left[u\frac{\partial^2 v}{\partial x^2} - u\frac{\partial^2 u}{\partial x \partial y} + v\frac{\partial^2 u}{\partial y^2} - v\frac{\partial^2 v}{\partial x \partial y}\right]\hat{k}$$

$$= \frac{\mu}{\rho}\left[\vec{\nabla}\times\frac{\partial^2 \vec{v}}{\partial x^2} + \vec{\nabla}\times\frac{\partial^2 \vec{v}}{\partial y^2}\right]$$

i.e. Considering $u = \frac{\partial \psi}{\partial y}$ and $v = -\frac{\partial \psi}{\partial y}$.

We got

$$\frac{\partial \psi}{\partial y}\frac{\partial}{\partial x}(\nabla^2 \psi) - \frac{\partial \psi}{\partial x}\frac{\partial}{\partial y}(\nabla^2\psi) = \frac{\mu}{\rho}\nabla^2(\nabla^2\psi)$$

We can define vorticity as $\nabla \times \vec{v}$ (curl of velocity vector)

i.e. Your momentum equation has only one dependent variable ψ.

It will be a fourth-order differential equation.

→ So you require four boundary conditions.

Note: For irrotational flow of inviscid fluids:

$$\vec{\nabla}\times\vec{v} = 0$$

i.e.

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \psi}{\partial y} & -\frac{\partial \psi}{\partial x} & 0 \end{vmatrix} = \hat{i}\left(-\frac{\partial^2\psi}{\partial x \partial z}\right) - \hat{j}\left(-\frac{\partial^2 \psi}{\partial y \partial z}\right) + \hat{k}\left(-\frac{\partial^2\psi}{\partial x^2} - \frac{\partial^2\psi}{\partial y^2}\right) = 0$$

i.e.

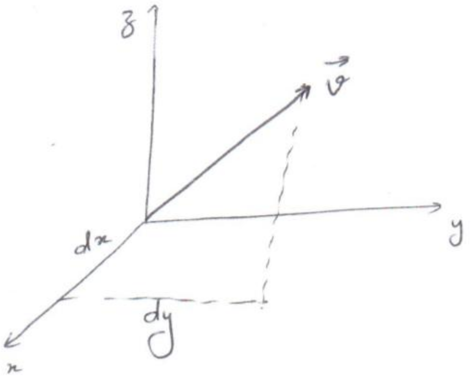
$$\boxed{\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0}$$

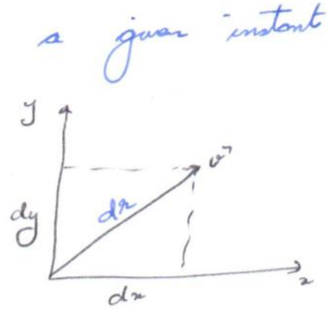
This is Laplace equation.

Q: What is the geometric interpretation of stream function ψ?

Recall the definition of streamlines.

A streamline is a line, which is tangent to velocity vector $\vec{v}$ everywhere at a given instant





$\Rightarrow$ Any elemental arc length $d\vec{r}$ of a streamline should be parallel to $\vec{v}$.

Therefore, their components should also match

The components should be such that:

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dr}{|\vec{v}|}$$

As $\quad u = \frac{\partial \psi}{\partial y}, \qquad v = -\frac{\partial \psi}{\partial x}$

$$\frac{dx}{u} = \frac{dy}{v}$$

" $\quad v\,dx - u\,dy = 0$

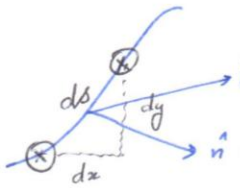
i.e. $\quad -\frac{\partial \psi}{\partial x}\,dx - \frac{\partial \psi}{\partial y}\,dy = 0$

or $\quad \frac{\partial \psi}{\partial x}\,dx + \frac{\partial \psi}{\partial y}\,dy = 0 = d\psi$ ~~$\Rightarrow$ a constant~~

i.e. ψ = a constant along a streamline

So in a flow domain, if we solve momentum equation for ψ, we can then subsequently plot streamlines or lines of constant ψ's.

Again consider a control surface



$\vec{v} = u\hat{i} + v\hat{j}$

$\hat{n} = \frac{dy}{ds}\hat{i} - \frac{dx}{ds}\hat{j}$

Now dQ between the two points across the control surface will be

$dQ = (\vec{v} \cdot \hat{n})\, dA$

Let us take unit width into the paper.

$\therefore \quad dA = ds \times 1 = ds$

i.e. $\quad dQ = (u\hat{i} + v\hat{j}) \cdot \left(\frac{dy}{ds}\hat{i} - \frac{dx}{ds}\hat{j}\right) ds \times 1$

$= \left(\frac{\partial\psi}{\partial y}\hat{i} - \frac{\partial\psi}{\partial x}\hat{j}\right)\left(\frac{dy}{ds}\hat{i} - \frac{dx}{ds}\hat{j}\right) ds$

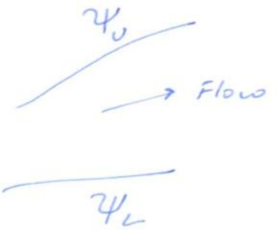
$= \frac{\partial\psi}{\partial x}dx + \frac{\partial\psi}{\partial y}dy = d\psi$

That is change in ψ across the element surface is equal to the volume flow through the element.

Volume flow between any two streamlines ① and ② $= \psi_2 - \psi_1$

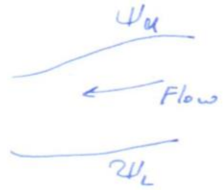
$= \int_1^2 dQ = \int_1^2 d\psi = \psi_2 - \psi_1$

Direction of flow. if $\psi_U > \psi_L$



If $\psi_U < \psi_L$

flow



Example

Find the stream function for following incompressible flow

$u = a(x^2 - y^2) \quad ; \quad v = -2axy \quad ; \quad w = 0.$