LECTURE 30
30-MAR-2016

ENERGY EQUATION AND DIFFERENT CONDITIONS

Yesterday, we started discussing on the Energy principle. From an elementary or differential volume $\Delta x \Delta y \Delta z$, we came up with the relation:

$$\mathring{Q} - \mathring{W}_v = \Delta x \Delta y \Delta z \left[\rho \frac{de}{dt} + \vec{v} \cdot \nabla p + p \nabla \cdot \vec{v} \right] \rightarrow (1)$$

$\Rightarrow$ Now we need to describe what all constitutes $\mathring{Q}$ and $\mathring{W}_v$ in equation (1).

To describe $\mathring{Q}$

We suggest that let heat transfer occurs only through conduction. We adopt ~~Fick's~~ Fourier's law of heat transfer.

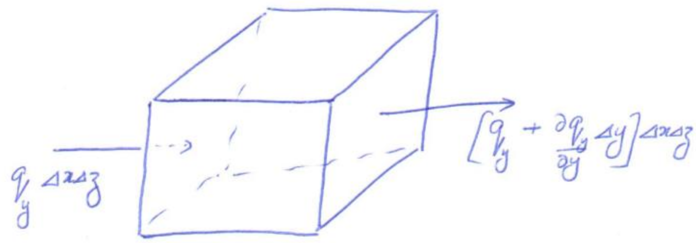
Let us define the heat flux or vector heat transfer per unit area as:

$$\vec{q} = -k \nabla T \quad ; \quad k \rightarrow \text{thermal conductivity}$$

Here $q_x = -k \frac{\partial T}{\partial x}$; $q_y = -k \frac{\partial T}{\partial y}$; $q_z = -k \frac{\partial T}{\partial z}$

Please note that this relation is shown with negative sign. This means that heat flux is positive in the direction of decreasing temperature.

→ For the six faces of the elementary rectangular volume



⇒ Let inlet fluxes per unit area

$q_x \Delta y \Delta z$; $q_y \Delta x \Delta z$; and $q_z \Delta x \Delta y$

and outlet fluxes be:

$\left[q_x + \frac{\partial}{\partial x}(q_x)\Delta x\right]\Delta y \Delta z$; $\left[q_y + \frac{\partial}{\partial y}(q_y)\Delta y\right]\Delta x \Delta z$; and

$\left[q_z + \frac{\partial}{\partial z}(q_z)\Delta z\right]\Delta x \Delta y$

∴ The net heat flux added to the system will be:

$$\dot{\Phi} = -\left[\frac{\partial}{\partial x}(q_x) + \frac{\partial}{\partial y}(q_y) + \frac{\partial}{\partial z}(q_z)\right]\Delta x \Delta y \Delta z$$

$$= -\nabla\cdot\vec{q}\ \Delta x \Delta y \Delta z$$

or $$\dot{\Phi} = \nabla\cdot(k\nabla T)\ \Delta x \Delta y \Delta z \longrightarrow (2)$$

To describe $\dot{W}_v$

The work done by the viscous forces can be described as follows:

Rate of change of viscous work component

= Viscous stress component * Velocity Component * Area of element face

For example, on left face of the elementary volume.

Let $\dot{W}_{v_{LF}} = w_y \, \Delta x \Delta z$

where $w_y = -\left(u\,\tau_{yx} + v\,\tau_{yy} + w\,\tau_{yz}\right)$

and $w_y \rightarrow$ Rate of change of viscous work per unit area on a plane perpendicular to y- direction

Similarly, let

$\dot{W}_{v_{RF}} \rightarrow$ i.e. on the right face

$$\dot{W}_{v_{RF}} = \left[w_y + \frac{\partial}{\partial y}(w_y)\,\Delta y\right]\Delta x \Delta z$$

In same way, you can describe viscous work rate in other direction planes as well.

$\therefore$ Net viscous work rate

$$\dot{W}_v = -\Big[\frac{\partial}{\partial x}\left(u\,\tau_{xx} + v\,\tau_{xy} + w\,\tau_{xz}\right) + \frac{\partial}{\partial y}\left(u\,\tau_{yx} + v\,\tau_{yy} + w\,\tau_{yz}\right) + \frac{\partial}{\partial z}\left(u\,\tau_{zx} + v\,\tau_{zy} + w\,\tau_{zz}\right)\Big]\Delta x \Delta y \Delta z$$

i.e. $$\dot{W}_v = -\nabla\cdot\left(\vec{v}\cdot\bar{\bar{\tau}}\right)\Delta x \Delta y \Delta z \rightarrow 3$$

Substituting equations 2 and 3 in equation 1, we get

$$\boxed{\nabla\cdot(k\nabla T) + \nabla\cdot\left(\vec{v}\cdot\bar{\bar{\tau}}\right) = \rho\frac{de}{dt} + \vec{v}\cdot\nabla p + p\,\nabla\cdot\vec{v}} \rightarrow 4$$

Boundary Conditions for differential Equations

You have studied in your mathematics courses that to solve the differential equations, you need to provide conditions like initial and boundary conditions.

⇒ This is very much true in fluid mechanics

Now the question is – how do you apply boundary condition based on the physical problem

⇒ Recall the three equations, we have developed:

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho \vec{v}) = 0$$

$$\rho \frac{d\vec{v}}{dt} = \rho \vec{g} - \nabla p + \nabla\cdot\overline{\overline{\tau}}$$

$$\nabla\cdot(k\nabla T) + \nabla\cdot(\vec{v}\cdot\overline{\overline{\tau}}) = \rho\frac{de}{dt} + \vec{v}\cdot\nabla p + p\,\nabla\cdot\vec{v}$$

where $e = \hat{u} + \frac{1}{2}V^2 + gz$

You can see that ρ, u, v, w, p, T are the dependent variables.

→ Based on the problem, one needs to apply

either
- Dirichlet boundary condition
- Neumann boundary condition
- Mixed - boundary conditions

Dirichlet boundary condition is where you specify the value of dependent variable at any location on the physical domain of the problem.

Neumann boundary condition is where you specify the gradient of the dependent variable at any location

⇒ If the problem is time-dependent, then you also need to specify initial conditions, i.e. the values of the dependent variable at all locations in the physical domain at initial time (say $t = 0$, etc.)

⇒ You may see that, for an impermeable wall, where there is no slip,

$$\vec{V}_{fluid} = \vec{V}_{wall} \quad (\text{mostly} = 0 \text{ for stationary walls})$$

(Dirichlet boundary condition).

⇒ For free surface or open channels, we may consider the free surface boundary as:

$$p_{free\ surface} = p_{atmospheric}$$

Vertical component of Velocity

$$w_{liquid} = w_{gas} \quad , \quad \text{etc.}$$

For incompressible flow,

Let us consider situations where μ, k, ρ are constants.

You will then have
$$\nabla \cdot \vec{v} = 0$$
$$\rho \frac{d\vec{v}}{dt} = \rho \vec{g} - \vec{\nabla} p + \mu \nabla^2 \vec{v}$$

You have $\vec{v}$ and p as unknowns

Solve for $\vec{v}$ and p.

For inviscid flow

$$\rho \frac{d\vec{v}}{dt} = \rho \vec{g} - \vec{\nabla} p$$

For inviscid flow; $\left(V_{normal}\right)_{fluid} = \left(V_{normal}\right)_{wall} = 0$ for fixed walls.