LECTURE 29
29 - MAR - 2016

DIFFERENTIAL APPROACH FOR ENERGY EQUATION

Yesterday we discussed the Navier - Stokes equation of fluid motion.

There we suggested that p, u, v, and w are dependent variables ~~and~~ in Navier - Stokes equations.

In an example problem, we asked what will be the condition or expression for pressure if $u = a(x^2 - y^2)$, $v = -2axy$, and $w = 0$ is solution for Navier - Stokes equation.

The problem was worked out in yesterday's lecture note.

⇒ Now let us briefly discuss about energy principle

Recall using Reynolds Transport Theorem, we had written the energy equation as:

$$\left.\frac{dE}{dt}\right|_{system} = \frac{d}{dt}\left[\int_{cv} e \rho \, dU\right] + \int_{cs} e \rho (\vec{V} \cdot \hat{n}) \, dA \quad \rightarrow (1)$$

where E = extensive property energy

e = intensive property $= \hat{u} + \frac{1}{2} V^2 + gz$

Subsequently equation (1) was written for a non-deformable fixed control volume as:

$$\frac{dQ}{dt} - \dot{W}_s - \dot{W}_v = \int_{cv} \frac{\partial}{\partial t}(e\rho) \, dU + \int_{cs} \left(e + \frac{p}{\rho}\right) \rho (\vec{V} \cdot \hat{n}) \, dA$$

Again as discussed in conservation of mass and linear momentum, we will take the same elementary or differential volume $\Delta x \Delta y \Delta z$

Recall what is $\dot{Q}$ → change in heat energy

$\dot{W}_s$ → shaft work change

$\dot{W}_v$ → rate of change of work due to viscous forces

⇒ For an elementary volume, there cannot be any infinitesimal shaft protruding its control volume (fixed Non-deform)

∴ We can consider $\dot{W}_s = 0$

Hence $$\dot{Q} - \dot{W}_v = \int_{cv} \frac{\partial}{\partial t}(e\rho)\,dU + \int_{cs}\left(e + \frac{p}{\rho}\right)\rho(\vec{v}\cdot\hat{n})\,dA$$

⇒ Adopt the same procedure done for mass and linear momentum: ~~(we also considering for infinitesimal element $\frac{\partial\rho}{\partial t} \approx 0$)~~

$$\therefore \dot{Q} - \dot{W}_v = \left[\frac{\partial}{\partial t}(\rho e) + \frac{\partial}{\partial x}\left(\rho u\left(e + \frac{p}{\rho}\right)\right) + \frac{\partial}{\partial y}\left(\rho v\left(e + \frac{p}{\rho}\right)\right) + \frac{\partial}{\partial z}\left(\rho w\left(e + \frac{p}{\rho}\right)\right)\right]\Delta x \Delta y \Delta z$$

i.e. $$\dot{Q} - \dot{W}_v = \Delta x \Delta y \Delta z\left[\rho\frac{\partial e}{\partial t} + \rho u\frac{\partial e}{\partial x} + \rho e\frac{\partial u}{\partial x} + u\frac{\partial p}{\partial x} + p\frac{\partial u}{\partial x} + ue\frac{\partial \rho}{\partial x} + ve\frac{\partial\rho}{\partial y} + \rho v\frac{\partial e}{\partial y} + \rho e\frac{\partial v}{\partial y} + v\frac{\partial p}{\partial y} + p\frac{\partial v}{\partial y} + e\frac{\partial\rho}{\partial t} + we\frac{\partial\rho}{\partial z} + \rho w\frac{\partial e}{\partial z} + \rho e\frac{\partial w}{\partial z} + w\frac{\partial p}{\partial z} + p\frac{\partial w}{\partial z}\right]$$

i.e. $$\dot{Q} - \dot{W}_v = \Delta x \Delta y \Delta z\left[\rho\frac{de}{dt} + \vec{v}\cdot\nabla p + p\,\nabla\cdot\vec{v}\right] \rightarrow (2)$$

Note that the term $e\left[\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w)\right] = 0$

Let us briefly work on the left hand side of the equation 2. That is, the terms $\dot{Q}$ and $\dot{W}$.

$\Rightarrow$ From Fick's law of heat transfer,

$\rightarrow$ The vector heat transfer per unit area

$$\vec{q} = -k \nabla T$$; where $k \rightarrow$ thermal conductivity

i.e. $q_x = -k \frac{\partial T}{\partial x}$; $q_y = -k \frac{\partial T}{\partial y}$; $q_z = -k \frac{\partial T}{\partial z}$ (another fluid property like viscosity).

Please note that this relation is shown with negative sign, which means that heat flux is positive in the direction of decreasing temperature. 'k' has units $J/s\,m\,K$.

$\Rightarrow$ Again for the ~~six~~ six faces of the rectangular element, let the inlet heat fluxes be:

$$q_x \Delta y \Delta z \ , \quad q_y \Delta x \Delta z \ , \quad q_z \Delta x \Delta y$$

and outlet heat fluxes be:

$$\left[q_x + \frac{\partial}{\partial x}(q_x)\Delta x\right]\Delta y \Delta z \ ; \ \left[q_y + \frac{\partial}{\partial y}(q_y)\Delta y\right]\Delta x \Delta z \ ;$$

and $$\left[q_z + \frac{\partial}{\partial z}(q_z)\Delta z\right]\Delta x \Delta y$$

$\therefore$ The net heat flux <u>added</u> to the system:

$$\dot{Q} = -\left[\frac{\partial}{\partial x}(q_x) + \frac{\partial}{\partial y}(q_z) + \frac{\partial}{\partial z}(q_z)\right]\Delta x \Delta y \Delta z$$

$$= -\nabla \cdot \vec{q} \ \Delta x \Delta y \Delta z$$

or $$\dot{Q} = \nabla \cdot (k \nabla T)\,\Delta x \Delta y \Delta z \longrightarrow 3$$

Rate of change of work done by viscous stresses

= Viscous Stress Component × Velocity Component × Area of element face

For example, on left face in y-direction

$$\dot{W}_{v_{LF}} = w_y \, \Delta x \Delta z \quad \text{, where}$$

$$w_y = -(u\,\tau_{yx} + v\,\tau_{yy} + w\,\tau_{yz})$$

On right face in y-direction

$$\dot{W}_{v_{RF}} = \left[w_y + \frac{\partial}{\partial y}(w_y)\,\Delta y \right] \Delta x \Delta z$$

∴ Again it is the gradient of 'w' that cause rate of change of viscous work. These viscous work fluxes explained with similar analogy as explained for heat energy. The outlet viscous work rate terms are deducted from inlet viscous work rate term, so as to get the <u>Net Viscous Work Rate</u>

$$\dot{W}_v = -\left[\frac{\partial}{\partial x}(u\,\tau_{xx} + v\,\tau_{xy} + w\,\tau_{xz}) + \frac{\partial}{\partial y}(u\,\tau_{yx} + v\,\tau_{yy} + w\,\tau_{yz}) + \frac{\partial}{\partial z}(u\,\tau_{zx} + v\,\tau_{zy} + w\,\tau_{zz}) \right] \Delta x \Delta y \Delta z$$

i.e. $$\dot{W}_v = -\,\nabla \cdot (\vec{v} \cdot \overline{\overline{\tau}})\,\Delta x \Delta y \Delta z$$

∴ Now the energy equation ② becomes:

$$\nabla \cdot (k \nabla T)\,\Delta x \Delta y \Delta z + \nabla \cdot (\vec{v} \cdot \overline{\overline{\tau}})\,\Delta x \Delta y \Delta z = \left[\rho \frac{de}{dt} + \vec{v} \cdot \nabla p + p\,\nabla \cdot \vec{v} \right] \Delta x \Delta y \Delta z$$

i.e. $$\nabla\cdot(k\nabla T) + \nabla\cdot(\vec{v}\cdot\bar{\bar{\tau}}) = \rho\frac{de}{dt} + \vec{v}\cdot\nabla p + p\,\nabla\cdot\vec{v} \rightarrow (4)$$

You can note that $e = \hat{u} + \frac{1}{2}V^2 + gz$.