LECTURE 28
28 - MARCH - 2016

NAVIER- STOKES EQUATIONS

In the last few classes, we were discussing on differential approach for linear momentum.

→ For a three - dimensional cartesian co-ordinate system, we came up with a system of partial differential equations describing the motion of fluid

$$\rho g_x - \frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\tau_{xx} + \frac{\partial}{\partial y}\tau_{yx} + \frac{\partial}{\partial z}\tau_{zx} = \rho\left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right]$$

$$\rho g_y - \frac{\partial p}{\partial y} + \frac{\partial}{\partial x}\tau_{xy} + \frac{\partial}{\partial y}\tau_{yy} + \frac{\partial}{\partial z}\tau_{zy} = \rho\left[\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right]$$

$$\rho g_z - \frac{\partial p}{\partial z} + \frac{\partial}{\partial x}\tau_{xz} + \frac{\partial}{\partial y}\tau_{yz} + \frac{\partial}{\partial z}\tau_{zz} = \rho\left[\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right]$$

⇒ These three set of PDE's is the most general form of equation of fluid motion that accounts for any type of fluid motion.

$$\boxed{\rho\vec{g} - \vec{\nabla}p + \nabla\cdot\overline{\overline{\tau}} = \rho\frac{d\vec{V}}{dt}}$$

⇒ If the fluid is INVISCID, then effects of viscosity are neglected. ∴ The equations of fluid motion reduces to

$$\boxed{\rho\vec{g} - \vec{\nabla}p = \rho\frac{d\vec{V}}{dt}}$$

↪ This is Euler's equation for <u>inviscid</u> fluid flow.

⇒ Recall our earlier discussions on viscosity.

We had defined Newton's law of viscosity. Newtonian liquids are those that obey linear law of viscosity.

For INCOMPRESSIBLE Newtonian fluids we can describe shear stresses in terms of velocity gradients. Recall ~~shear~~ viscous stress tensor

$$\overline{\overline{\tau}} = \begin{pmatrix} \tau_{xx} & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \tau_{yy} & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \tau_{zz} \end{pmatrix}$$

$\Rightarrow$ Incompressible Liquids

Here for Newtonian fluids, $\tau_{xx} = 2\mu \frac{\partial u}{\partial x}$

$$\tau_{yx} = \tau_{xy} = \mu\left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right]$$

$$\tau_{zx} = \tau_{xz} = \mu\left[\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right]$$

$$\tau_{yy} = 2\mu \frac{\partial v}{\partial x}, \quad \tau_{zz} = 2\mu \frac{\partial w}{\partial z}$$

$\therefore$ The equations of motion will then be for constant-density or incompressible fluid.

$$\rho g_x - \frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left[2\mu \frac{\partial u}{\partial x}\right] + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\right]$$
$$= \rho\left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right]$$

i.e. $$\rho g_x - \frac{\partial p}{\partial x} + \mu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right] + \mu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 w}{\partial x \partial z}\right]$$
$$= \rho\left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right]$$

[Ofcourse, we are assuming μ is a constant] i.e.

$$\rho g_x - \frac{\partial p}{\partial x} + \mu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right] + \mu \frac{\partial}{\partial x}\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right]$$
$$= \rho\left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right]$$

Recall for incompressible fluid, what is $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$?!

∴ We get

$$\boxed{\begin{aligned}
&\rho g_x - \frac{\partial p}{\partial x} + \mu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right] = \rho\left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right]\\
&\text{My}\quad \rho g_y - \frac{\partial p}{\partial y} + \mu\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right] = \rho\left[\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right]\\
&\text{and}\quad \rho g_z - \frac{\partial p}{\partial z} + \mu\left[\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right] = \rho\left[\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right]
\end{aligned}} \Rightarrow \text{II}$$

This set of three PDE's is the famous NAVIER-STOKES equations of fluid motion for incompressible Newtonian fluids.

Navier-Stokes equations are second-order non-linear partial differential equations. You can solve them for various cases. Also you can see that p, u, v, w are the unknowns or dependent variables in the above expressions. You have to use the equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

as fourth one, while solving them simultaneously.

⇒ The computational fluid dynamics (CFD) solves these partial differential equations while modeling the fluid flow.

Example (As adopted from FM WHITE)

It was observed that velocity field for a fluid flow was obtained as: $u = a(x^2-y^2)$, $v = -2axy$, $w = 0$. You know u, v, w, and p are solutions of Navier-Stokes equations. Now find under what condition the above expressions for u, v, w will be solutions of Navier-Stokes equations. Consider $g_x = g_y = 0$ and $g_z = -g$.

~~Solution~~ Answer

⇒ ~~As~~ it is given

$u = a(x^2-y^2)$, $\therefore \frac{\partial u}{\partial x} = 2ax$

$v = -2axy$, $\frac{\partial u}{\partial y} = -2ay$

$w = 0$, $\frac{\partial v}{\partial x} = -2ay$, $\frac{\partial v}{\partial y} = -2ax$

i.e. They are not functions of time 't'.

Hence the flow will be steady.

⇒ We need to find $p(x, y, z)$

$$\Rightarrow \rho * 0 - \frac{\partial p}{\partial x} + \mu\left[2a - 2a + 0\right] = \rho\left[\cancelto{0}{\frac{\partial u}{\partial t}} + a(x^2-y^2)\,2ax + (-2axy)(-2ay) + 0\right]$$

$$\rho * 0 - \frac{\partial p}{\partial y} + \mu\left[0+0+0\right] = \rho\left[\cancelto{0}{\frac{\partial v}{\partial t}} + a(x^2-y^2)(-2ay) + (-2axy)(-2ax) + 0\right]$$

$$\rho * -g - \frac{\partial p}{\partial z} + \mu * 0 = \rho\left[0\right]$$

i.e.

$$-\frac{\partial p}{\partial x} = \rho\left[2a^2x^3 - 2a^2xy^2 + 4a^2xy^2\right]$$

$$-\frac{\partial p}{\partial y} = \rho\left[-2a^2x^2y + 2a^2y^3 + 4a^2x^2y\right]$$

$$\left.\begin{aligned} \frac{\partial p}{\partial z} &= -\rho g \\ \frac{\partial p}{\partial x} &= -\rho\, 2a^2 x\,(x^2+y^2) \\ \frac{\partial p}{\partial y} &= -\rho\, 2a^2 y\,(x^2+y^2) \end{aligned}\right\}$$

So vertical pressure gradient is hydrostatic.

Pressure do vary in the x-y plane.

Here note that $\frac{\partial}{\partial x}\left(\frac{\partial p}{\partial y}\right) = -4\rho a^2 y x$

and $\frac{\partial}{\partial y}\left(\frac{\partial p}{\partial x}\right) = -4\rho a^2 x y$

Both are same.

$\therefore$ The solutions u, v, and w are exact solutions of Navier-Stokes equation.

$\Rightarrow$ The solution p will be.

$$p = \int \frac{\partial p}{\partial x}\, dx \Big|_{y,z} = \int -2\rho a^2 x (x^2+y^2)\, dx$$

$$= -2a^2\rho\left[\frac{x^4}{4} + \frac{x^2y^2}{2}\right] + f_1(y,z)$$

Again now, $\frac{\partial p}{\partial y} = -2a^2\rho\, x^2 y + \frac{\partial f_1}{\partial y} = -2\rho a^2 y(x^2+y^2)$

$$\therefore \frac{\partial f_1}{\partial y} = -2a^2\rho y^3$$

$$\text{or } f_1 = \int \frac{\partial f_1}{\partial y}\, dy\Big|_z = -2a^2\rho \frac{y^4}{4} + f_2(z)$$

$$\therefore p = -2a^2\rho\left[\frac{x^4}{4} + \frac{x^2y^2}{2}\right] - 2a^2\rho\frac{y^4}{4} + f_2(z)$$

$$\therefore \frac{\partial p}{\partial z} = 0 + \frac{\partial f_2}{\partial z} = -\rho g$$

a $f_2(z) = -\rho g z + C$

where C is a constant

$$\therefore \quad p(x, y, z) = -2a^2\rho\left[\frac{x^4}{4} + \frac{y^4}{4} + \frac{x^2y^2}{2}\right] - \rho g z + C$$