LECTURE 26

16-MAR-2016

LINEAR MOMENTUM IN DIFFERENTIAL APPROACH (CONTD...)

Yesterday, we started discussing about linear momentum principle through differential approach.

We have seen for an elemental volume $\Delta x \Delta y \Delta z$

$$\Sigma \vec{F} = \rho \, \Delta x \Delta y \Delta z \left[\frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \nabla) \vec{V} \right]$$

or $$\Sigma \vec{F} = \rho \frac{d\vec{V}}{dt} \Delta x \Delta y \Delta z$$

$\Rightarrow$ There will be - Body forces and surface forces that contribute to $\Sigma \vec{F}$

→ For this situation, let us consider the body force is only due to gravity

$$\therefore \quad d\vec{F}_{grav} = \rho \vec{g} \, \Delta x \Delta y \Delta z$$

where $$\vec{g} = 0\hat{i} + 0\hat{j} - g\hat{k}$$

→ The surface force will be due to pressure and viscous stresses.

→ The surface forces can be related with respect to the stresses on the sides of the rectangular element considered here.

These stresses are sum of hydrostatic pressure and viscous stress.

Viscous stresses can be represented as $\overline{\overline{\tau}}$ or in short form τ_{ij}. They are tensorial quantities.

e.g. Velocity is a vector

$$\vec{v} = u\hat{i} + v\hat{j} + w\hat{k} \rightarrow v_i$$

i.e. It ~~has~~ has three componenents.

Or in other words, we require three components to define a vector.

Similarly a second rank tensorial quantity requires nine components to be described.

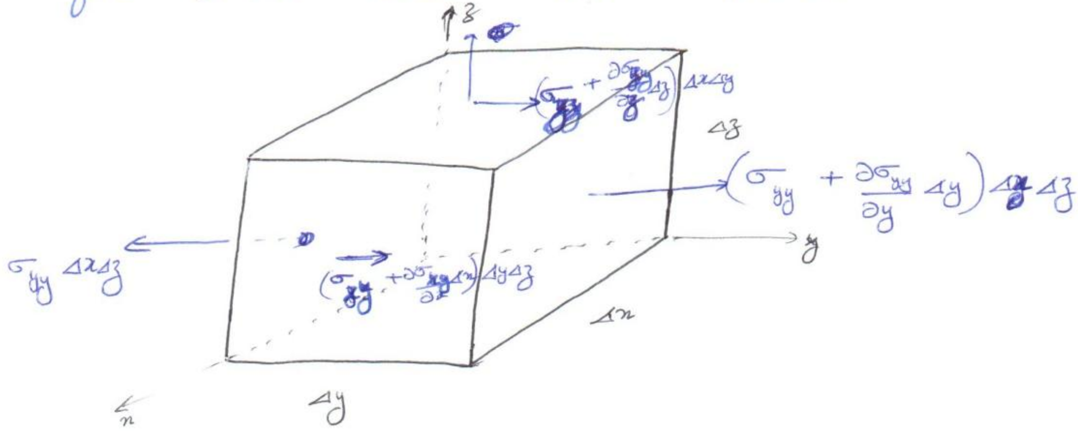
$$\overline{\overline{\tau}} = \begin{pmatrix} \tau_{xx} & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \tau_{yy} & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \tau_{zz} \end{pmatrix} \rightarrow \tau_{ij}$$

Similarly, strain rates ε_{ij} are also second rank tensors.

The stress, therefore, is a second rank tensor

$$\overline{\overline{\sigma}} \rightarrow \sigma_{ij} = \begin{pmatrix} -p + \tau_{xx} & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & -p + \tau_{yy} & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & -p + \tau_{zz} \end{pmatrix}$$

As discussed earlier for pressure, here also, it is the gradients or differences that cause a net surface force on the elemental control volume.



Consider ~~again~~ the surface force, right now, only in y- direction

You have normal forces $\left\{\left(\sigma_{yy} + \frac{\partial \sigma_{yy}}{\partial y}\Delta y\right)\Delta x \Delta z - \sigma_{yy}\,\Delta x \Delta z\right\}$

$$= \frac{\partial \sigma_{yy}}{\partial y}\Delta x \Delta y \Delta z$$

You have also tangential or shear forces in y- direction

$$\left(\sigma_{xy} + \frac{\partial \sigma_{xy}}{\partial x}\Delta x\right)\Delta y \Delta z - \sigma_{xy}\,\Delta y \Delta z$$

and $\left(\sigma_{zy} + \frac{\partial \sigma_{zy}}{\partial z}\Delta z\right)\Delta x \Delta y - \sigma_{zy}\,\Delta x \Delta y$

∴ The Net surface force in x - direction

$$dF_{y,\,surface} = \left[\frac{\partial(\sigma_{xy})}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{zy}}{\partial z}\right]\Delta x \Delta y \Delta z$$

$$\frac{dF_{y,\,surface}}{dV} = -\frac{\partial p}{\partial y} + \frac{\partial}{\partial x}(\tau_{xy}) + \frac{\partial}{\partial y}(\tau_{yy}) + \frac{\partial}{\partial z}(\tau_{zy})$$

||ly
$$\frac{dF_{x,\,surface}}{\Delta x \Delta y \Delta z} = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial x}(\tau_{xx}) + \frac{\partial}{\partial y}(\tau_{yx}) + \frac{\partial}{\partial z}(\tau_{zx})$$

$$\frac{dF_{z,\,surface}}{\Delta x \Delta y \Delta z} = -\frac{\partial p}{\partial z} + \frac{\partial}{\partial x}(\tau_{xz}) + \frac{\partial}{\partial y}(\tau_{yz}) + \frac{\partial}{\partial z}(\tau_{zz})$$

$$dF_{x_{surf}}\,\hat{i} + dF_{y_{surf}}\,\hat{j} + dF_{z_{surf}}\,\hat{k} = d\vec{F}_{surface}$$

$$\frac{d\vec{F}_{surface}}{\Delta x \Delta y \Delta z} = -\nabla p + \frac{d\vec{F}_{viscous}}{\Delta x \Delta y \Delta z}$$

Now
$$\frac{d\vec{F}_{viscous}}{\Delta x \Delta y \Delta z} = \hat{i}\left[\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right] + \hat{j}\left[\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}\right] + \hat{k}\left[\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z}\right]$$

$$= \nabla \cdot \bar{\bar{\tau}}$$

, please note $\bar{\bar{\tau}}$ is a second rank tensor

$$\bar{\bar{\tau}} = \begin{pmatrix} \tau_{xx} & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \tau_{yy} & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \tau_{zz} \end{pmatrix}$$

$\overline{\overline{\tau}}$ → Viscous stress tensors acting on the elemental fluid volume $\Delta x \Delta y \Delta z$.

∴ We have now from the relation:

$$\sum \vec{F} = \rho \frac{d\vec{v}}{dt} \Delta x \Delta y \Delta z$$

i.e. $$d\vec{F}_{grav} + d\vec{F}_{surface} = \rho \frac{d\vec{v}}{dt} \Delta x \Delta y \Delta z$$

i.e. $$\rho \vec{g} \Delta x \Delta y \Delta z + \left[(-\nabla p) \Delta x \Delta y \Delta z + \nabla \cdot \overline{\overline{\tau}} \Delta x \Delta y \Delta z\right] = \rho \frac{d\vec{v}}{dt} \Delta x \Delta y \Delta z$$

As the volume is arbitrary, the equation becomes expression at any mathematical point:

$$\boxed{\rho \vec{g} - \nabla p + \nabla \cdot \overline{\overline{\tau}} = \rho \frac{d\vec{v}}{dt}} \longrightarrow (1)$$

$$\rho g_x - \frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = \rho \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right)$$

$$\rho g_y - \frac{\partial p}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} = \rho \left[\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \right]$$

$$\rho g_z - \frac{\partial p}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} = \rho \left[\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \right]$$