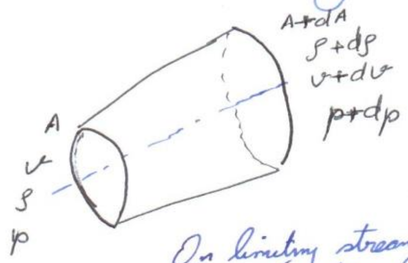
LECTURE 21

07-MARCH-2016

Bernoulli's Principle through Linear Momentum Principle

Recall, in the last class, we were discussing about a streamtube.

An elemental streamtube



On limiting streamtube becomes streamline

→ The linear momentum equation in the streamwise direction 's' will be:

$$\Sigma\, dF_s = \frac{d}{dt}\left[\int_{cv} \vec{v}\, \rho\, dU\right] + \left[\dot{m}\, v\right]_{out} - \left[\dot{m}\, v\right]_{inlet}$$

Also recall from conservation of mass, we have seen

$$d\dot{m} = d(\rho A v) = -\frac{\partial \rho}{\partial t} A\, ds$$

Again

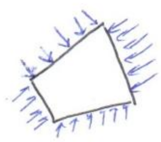
$$\Sigma\, dF_s \approx \frac{\partial}{\partial t}(\rho v)\, A\, ds + d(\dot{m} v)$$

Note that v_s = velocity in streamline direction = v

The force components, $dF_{s,\,grav} = -dW \sin\theta$
$= -\rho g A\, ds \sin\theta$

$$dF_{s,\,press} = pA - (p + dp)(A + dA) + \frac{1}{2}\, dp\, dA$$

$$\approx -A\, dp$$



$$\therefore \quad \sum dF_s = dF_{s,\,grav} + dF_{s,\,pres}$$

$$= -\rho g A\, ds \sin\theta - A\,dp$$

i.e. $$-\rho g A\, ds \sin\theta - A\,dp = \frac{\partial}{\partial t}(\rho v)\, A ds + d(\dot{m} v)$$

$$= \frac{\partial \rho}{\partial t} v A\, ds + \frac{\partial v}{\partial t} \rho A\, ds + \dot{m}\, dv + v\, d\dot{m}$$

As $d\dot{m} = -\frac{\partial \rho}{\partial t} A\, ds$, and $\dot{m} = \rho A v$

$$\therefore \quad -\rho g A\, dz - A\, dp = v\left(\frac{\partial \rho}{\partial t} A\, ds + d\dot{m}\right) + \frac{\partial v}{\partial t} \rho A\, ds + \rho A v\, dv$$

i.e. $$\boxed{\frac{\partial v}{\partial t} ds + v\, dv + g\, dz + \frac{dp}{\rho} = 0}$$

This is Bernoulli's equation for unsteady frictionless flow along a streamline.

On integrating this differential equation between any two points ① and ② on a streamline, we get

$$\int_1^2 \frac{\partial v}{\partial t} ds + \int_1^2 \frac{dp}{\rho} + \frac{1}{2}(v_2^2 - v_1^2) + g(z_2 - z_1) = 0$$

For steady incompressible flow,

$\frac{\partial v}{\partial t} = 0$, and ρ = constant

$$\therefore \quad \frac{(p_2 - p_1)}{\rho} + \frac{1}{2}(v_2^2 - v_1^2) + g(z_2 - z_1) = 0$$

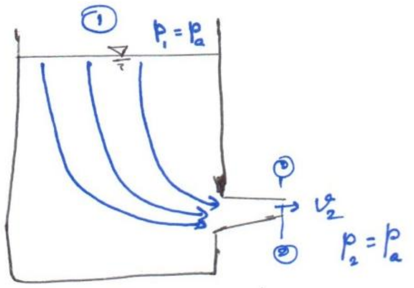
or

$$\boxed{\frac{p_2}{\rho} + \frac{v_2^2}{2} + g z_2 = \frac{p_1}{\rho} + \frac{v_1^2}{2} + g z_1 = \text{Constant}}$$

Please note that this constant can differ for different streamlines.

Example.

Find a relation between nozzle discharge velocity v_2 and tank free surface height 'h', if the flow is assumed steady and frictionless.



Solution.

For frictionless, steady flow along a streamline

$$\frac{p}{\rho} + \frac{v^2}{2} + gz = \text{a constant}$$

i.e. $$\frac{p}{\rho g} + \frac{v^2}{2g} + z = \text{constant} = \text{Total Head}$$

As there is no friction loss, this total head will be a constant.

In the figure, section 1 is upstream, section 2 is nozzle.

$$p_1 = p_a = \text{atmospheric pressure}$$

$$p_2 = p_a = \text{atmospheric pressure.}$$

$\therefore$ ~~Along a streamline~~ At the upstream,

$$v_1 = 0$$

$$v_2 = ??$$

Total Head at u/s $= \dfrac{p_1}{\rho g} + \dfrac{v_1^2}{2g} + z_1 = \text{constant} = H$

$$\text{i.e. } H = \frac{p_a}{\rho g} + z_1$$

The total head will be same at section 2.

$$\text{i.e. } \frac{p_1}{\rho} + \frac{v_1^2}{2} + g z_1 = \frac{p_2}{\rho} + \frac{v_2^2}{2} + g z_2$$

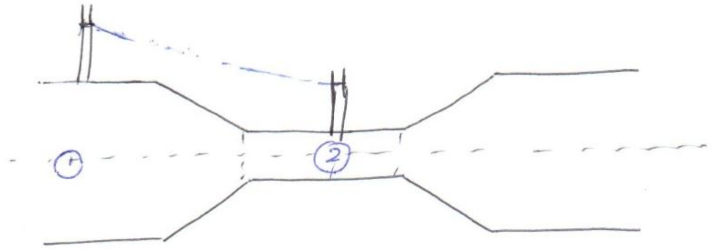
$v_1 = 0$, $p_1 = p_2 = p_a$, ρ is same.

$$\therefore \frac{v_2^2}{2} = g(z_1 - z_2)$$

$$\text{or } v_2^2 = 2g(z_1 - z_2)$$

$$\text{or } \underline{\underline{v_2^2 = 2gh}}$$, where $h = z_1 - z_2$

Example: (Adopted from FM White).
A venturitube is used to measure pressure difference. Find an expression for mass flux in the tube as a function of pressure change.



Considering the flow to be steady, frictionless, we can use Bernoulli's principle.

Again presuming same streamline pass through centre of section 1 and section 2

$$\frac{P_1}{\rho} + \frac{v_1^2}{2} + g z_1 = \frac{p_2}{\rho} + \frac{v_2^2}{2} + g z_2$$

Since tube is horizontal, $z_1 = z_2$.

$$v_2^2 - v_1^2 = \frac{2}{\rho}(P_1 - P_2)$$

As $A_1 v_1 = A_2 v_2$

we have, $v_1 = \frac{A_2}{A_1} v_2$.

If the section is circular, $A_1 = \frac{\pi}{4} D_1^2$

$A_2 = \frac{\pi}{4} D_2^2$

$$\therefore \quad v_1 = \left(\frac{D_2}{D_1}\right)^2 v_2$$

$$\therefore \quad v_2^2 - \left(\frac{D_2}{D_1}\right)^4 v_2^2 = \frac{2}{\rho}(P_1 - P_2)$$

or $$v_2^2\left(1 - \frac{D_2^4}{D_1^4}\right) = \frac{2}{\rho}(P_1 - P_2)$$

Mass flux is $\quad \dot{m} = \rho A v$

$$\dot{m} = \rho_0 A_1 v_1 = \rho_0 A_2 v_2$$

$$= \rho A_2 \left[\frac{\frac{2}{\rho}(p_1 - p_2)}{\left(1 - \frac{D_2^4}{D_1^2}\right)} \right]^{1/2}$$