LECTURE 15

16 - FEB - 2016

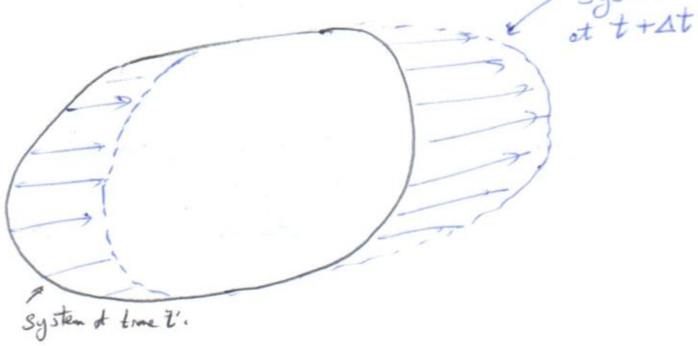
REYNOLDS TRANSPORT THEOREM (CONTD...)

Yesterday, we started discussing on Reynolds Transport Theorem. For application of conservation laws – mass, linear momentum, angular momentum, energy, etc. – which can be directly applied for a system (as in solid mechanics), you may find it difficult in fluid mechanics. Because inside the control volume, the fluid system is changing.

We need to convert system analysis to control volume analysis that is represented by Reynolds Transport Theorem.

⇒ We had discussed about Volume Flow Rate and Mass Flow Rate.

⇒ While describing RTT, we drew an arbitrary control volume.



→ The black color solid line is a control volume of fluid

→ At an instant 't' a system of fluid occupies the space inside the control volume.

→ We have a fixed volume in the space and it is called control volume.

→ However, the system that occupied the control volume at time 't' might have moved to a different position at time '$t + \Delta t$'.

→ Recall about the extensive property B and intensive property β.

$$B_{cv} = \int_{cv} \beta \rho \, dU$$

⇒ There can be three reasons for change of property B in the control volume

(i) The time rate of change of B within the control volume → $\frac{d}{dt}\left(\int_{cv} \beta \rho \, dU\right)$

(ii) The outflow of property B through the control surfaces of the volume → $\int_{cs} \beta \rho \; v \cos\theta \; dA_{out}$

(iii) The inflow of property B through the control surfaces of the volume → $\int_{cs} \beta \rho \; v \cos\theta \; dA_{in}$

⇒ Both the outflow and inflow can be marked as <u>net outflow</u>. Also note that through the surface, where inflow occurs, $\vec{v} \cdot \hat{n}$ is always <u>negative</u>.

Similarly, $\vec{v}\cdot\hat{n}$ is positive, where outflow occurs.

$\therefore$ The changes in B in control volume can be summated as:
$$\frac{d}{dt}\left[\int_{cv} \beta \rho \, dU\right] + \int_{cs} \beta \rho \, \vec{v}\cdot\hat{n} \, dA$$

$\Rightarrow$ If we tend $\Delta t \to 0$, then the control volume will be same as the system volume at time 't'.

That is, we can relate the time rate of change of property B stored in system with respect to that of control volume.

$\therefore$ When $\Delta t \to 0$

$$\boxed{\frac{d}{dt}\left(B_{system}\right) = \frac{d}{dt}\left[\int_{cv} \beta \rho \, dU\right] + \int_{cs} \beta \rho \, \vec{v}\cdot\hat{n} \, dA}$$

This is Reynolds Transport Theorem.

$\Rightarrow$ If we have a fixed control volume, which is non-deformable, then we can write RTT as:

$$\frac{d}{dt}\left(B_{system}\right) = \int_{cv} \frac{\partial}{\partial t}(\beta \rho) \, dU + \int_{cs} \beta \rho \, (\vec{v}\cdot\hat{n}) \, dA$$

$\Rightarrow$ If the control volume is moving at a constant velocity, $\vec{V_s}$, then we can define relative velocity $\vec{v_r} = \vec{v} - \vec{V_s}$ and your RTT will be:

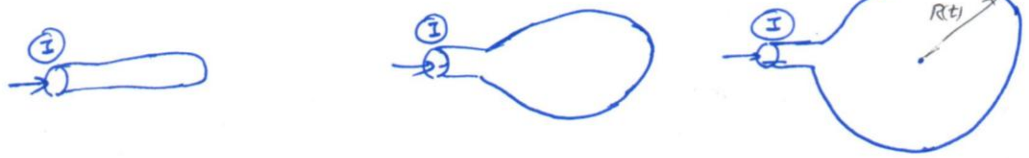
$$\frac{d}{dt}\left(B_{system}\right) = \frac{d}{dt}\left[\int_{cv} \beta \rho \, dU\right] + \int_{cs} \beta \rho \, (\vec{v_r}\cdot\hat{n}) \, dA$$

Example (As adopted from FM White's Fluid Mechanics)

A baloon is filled through section I. Area at that section is A_I, velocity is V_I, fluid density is ρ_I.
The average density within the baloon is $\rho_b(t)$. Find an expression for rate of change of fluid system mass within the baloon at this instant.

Solution

You know that the baloon volume changes with respect to time.



→ For this case, the baloon boundary forms the control volume. However, this volume increases with time.

→ This example is a case of deformable control volume

We will use the RTT

$$\frac{d}{dt}\left(B_{system}\right) = \frac{d}{dt}\left[\int_{cv} \beta \rho \, dV\right] + \int_{cs} \beta \rho \, \vec{V}_r \cdot \hat{n} \, dA$$

B_{system} = Mass of air in the baloon = m_{system}

∴ From conservation of mass principle,

$$\frac{d}{dt}\left[m_{system}\right] = 0.$$

Also $\beta = 1$

$\therefore$ In RTT,

$$0 = \frac{d}{dt}\left[\int_{cv} \rho \, dU\right] + \int_{cs} \rho (\vec{V_r} \cdot \hat{n}) \, dA$$

At inlet, $|\vec{V_r}| = V_I$

At outlet, $|\vec{V_r}| = 0$

There is only one inlet portion and there is no outflow section.

$$\therefore \quad 0 = \frac{d}{dt}\left[\int_{cv} \rho_b(t) \, dU\right] + \int_{outlet} \rho * 0 - \int_{inlet} \rho (\vec{V_r} \cdot \hat{n}) \, dA$$

$$0 = \frac{d}{dt}\left[\rho_b \frac{4\pi}{3} R^3\right] - \rho_I A_I V_I$$

or $$\frac{d}{dt}(\rho_b R^3) = \frac{3}{4\pi} \rho_I A_I V_I$$

Application of RTT ~~for~~ to conservation of mass principle

As the general form of RTT is:

$$\frac{d}{dt}(B_{system}) = \frac{d}{dt}\left[\int_{cv} \beta \rho \, dU\right] + \int_{cs} \beta \rho (\vec{V_r} \cdot \hat{n}) \, dA$$

For conservation of mass, $B_{system} = m_{system}$

B = mass of fluid in control volume at instant 't'.

$\beta = 1$, $\therefore B_{cv} = \int_{cv} \rho \, dU$

$$\frac{d m_{system}}{dt} = 0 = \frac{d}{dt}\left[\int_{cv} \rho \, dU\right] + \int_{cs} \rho (\vec{V_r} \cdot \hat{n}) \, dA$$

For a fixed, non-deformable control volume,

$$0 = \int_{cv} \frac{\partial}{\partial t}\left[\rho \, dU\right] + \int_{cs} \rho(\vec{V}\cdot\hat{n})\, dA$$

i.e. $$0 = \int_{cv} \frac{\partial \rho}{\partial t}\, dU + \int_{cs} \rho(\vec{V}\cdot\hat{n})\, dA$$

⇒ If the flow is <u>steady</u>, then

$$\int_{cs} \rho(\vec{V}\cdot\hat{n})\, dA = 0$$

That is net outflow is zero. (Or Inflow and Outflow are equal).

If control surface has one-dimensional inlets and outlets

Then $$\sum_{\substack{\text{inlet}\\\text{section}}} A_i V_i = \sum_{\substack{\text{outlet}\\\text{section}}} A_j V_j$$

⇒ Incompressible liquids:

An incompressible liquid is one in which the density does not vary significantly.

∴ In a non-deformable fixed control volume:

$$0 = \int_{cv} \frac{\partial \rho}{\partial t}\, dU + \int_{cs} \rho(\vec{V}\cdot\hat{n})\, dA$$

$\frac{\partial \rho}{\partial t} = 0$ (due to incompressibility).

$$\int_{cs} \rho(\vec{V}\cdot\hat{n})\, dA = 0$$