LECTURE 14

15-FEB-2016

REYNOLDS TRANSPORT THEOREM

* In the last lecture, we described about control volumes.
* The difference between a fluid system and a control volume was also described.
* You are also that aware that the conservation principles on mass, linear momentum, angular momentum, energy, etc. can be directly applied to a fluid system.

Q: Then how these principles can be described when you adopt control volume approach?

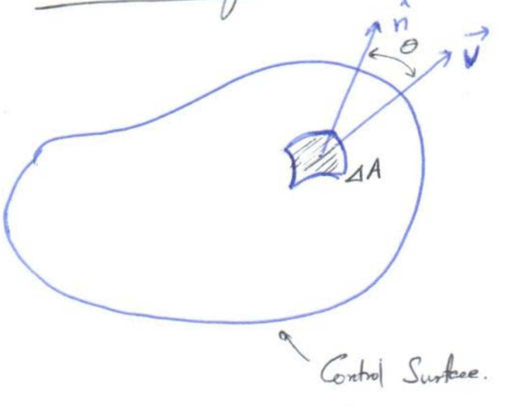
Ans: If you take a flowing fluid, control volume, you know that the fluid system that was initially occupying the control volume will be replaced by another fluid system at the next instant.

To convert the system analysis conservation concept to control volume conservation concept, we need to appropriately relate both mathematically and conceptually.

The conversion from system analysis to control volume analysis is represented by <u>Reynolds Transport Theorem</u>.

⇒ As described in the last class that a <u>system</u> is some fixed property described in space. It has surroundings and is separated from surroundings by boundaries.

Similarly a control volume, we can envisage as a, volume occupying a space and have some shape. The volume consists of surfaces called <u>control surfaces</u>.



→ If you take a small elemental area on the control surface, ΔA

→ It has an outward normal unit vector $\hat{n}$.

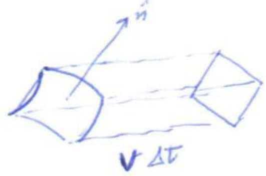
→ Let the velocity vector of fluid passing that elemental area be $\vec{v}$

→ There are chances that $\hat{n}$ and $\vec{v}$ may not be collinear.

→ The volume of fluid that will sweep through this elemental area ~~will be~~ in an elemental time Δt will be:

$$\Delta V = v \, dt \, \Delta A \cos\theta$$

i.e. $\Delta V = (\vec{v} \cdot \hat{n}) \Delta A \Delta t$



This is the component of velocity vector in the direction of $\hat{n}$. or compont of ~~velocity~~ area vector in the direction of $\vec{V}$.

$$\therefore \frac{\Delta V}{\Delta t} = (\vec{v} \cdot \hat{n}) \Delta A$$

~~As~~ ~~at~~ ΔA on integrating through the control surface will give total area of control surface S.

$\frac{\Delta V}{\Delta t}$ was the volume flow rate through the elemental area ΔA.

To get the total volume rate of flow Q through S, we will limit the quantities $\lim_{\substack{\Delta t \to 0 \\ \Delta A \to 0}} \frac{\Delta V}{\Delta t} = \frac{dV}{dt}$ ~~all~~

and $$Q = \int_S \frac{dV}{dt} = \int_S (\vec{v} \cdot \hat{n}) \, dA$$

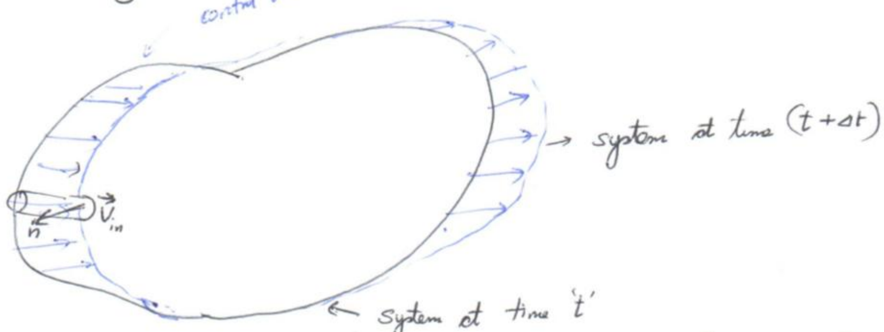
$Q \rightarrow$ Total volume flow rate.

$\Rightarrow$ If the fluid concerned in the control volume has a density ρ.

Then mass flow rate

$$\dot{m} = \int_S \rho (\vec{v} \cdot \hat{n}) \, dA$$

⇒ To describe RTT, let us first take an arbitrary fixed control volume.



→ Initially the control volume is filled by fluid of certain property and we can term that as system property.

→ There will be inflow and outflow of fluid through the control surfaces. For that we need to take into account the earlier mentioned volume flow rate and mass flow rate, etc.

→ Let B be any property of the fluid (e.g. Mass, momentum, energy, etc.)

* Let $\beta = \dfrac{dB}{dm}$ be suggested as an intensive property or the amount of B per unit mass in any small element of the fluid.

B is now called the extensive property.

→ Initially the control volume is same as system volume.

$$B_{cv} = \int_{cv} \beta \, dm = \int_{cv} \beta \rho \, dU$$

Please note that here, I am using ~~the~~ dU as the integral variable (instead of dV).

→ If 't' is the beginning time, then the black color bold line denotes the system boundary and interprets the system volume. This volume will be same as the control volume.

→ After a certain time 't+Δt', the ~~syst~~ same system may be occupying the space as shown in blue line.

→ Therefore, in the control volume, there can be three changes for the property B.

(i) A change within the control volume

$$\rightarrow \frac{d}{dt}\left(\int_{cv} \beta \rho \, dU\right)$$

(ii) Outflow of property B from the control volume

$$\rightarrow \int_{cs} \beta \rho \, v \cos\theta \, dA_{out}$$

(iii) Inflow of property B to the control volume

$$\rightarrow \int_{cs} \beta \rho \, v \cos\theta \, dA_{in}$$

→ You can see that for inflow to the control volume through the control surface the term $\vec{v} \cdot \hat{n}$ should be negative. For outflow, $\vec{v} \cdot \hat{n}$ should be positive.

When $\Delta t \to 0$, the system volume and control volume can be compared.

That is $\Delta t \to 0$, the instantaneous change of property B in the system is same as change of property B in the control volume.

i.e. $$\frac{d}{dt}\left(B_{system}\right) = \frac{d}{dt}\left(\int_{cv} \beta \rho \, dU\right) + \int_{cs} \beta \rho \, v \cos\theta \, dA_{out} - \int_{cs} \beta \rho \, v \cos\theta \, dA_{in}$$

This is Reynolds Transport theorem.

This explanation was for an arbitrary fixed control volume.

We can also write

$$\boxed{\frac{d}{dt}\left(B_{system}\right) = \frac{d}{dt}\left(\int_{cv} \beta \rho \, dU\right) + \int_{cs} \beta \, \rho (\vec{V} \cdot \hat{n}) \, dA}$$

Note:

Earlier, we talked about a fixed control volume. In a fixed control volume, the volume elements are not changing.

$$\therefore \frac{d}{dt}\left[\int_{cv} \beta \rho \, dU\right] = \int_{cv} \frac{\partial}{\partial t}(\beta \rho) \, dU$$

Note: 2) If the control volume is moving at a constant velocity $\vec{V}_s$,
then we can define relative velocity $\vec{V}_r = \vec{V} - \vec{V}_s$
where $\vec{V}$ is the actual fluid velocity.

$$\frac{d}{dt}\left(B_{system}\right) = \frac{d}{dt}\left[\int_{CV} \beta \rho \, dU\right] + \int_{CS} \beta \rho \left(\vec{V}_r \cdot \hat{n}\right) dA$$

For Arbitrarily Moving & Deformable Control Volume

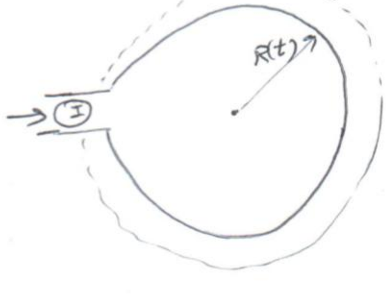
If the control volume is moving and deforming then

$$\frac{d}{dt}\left(B_{system}\right) = \frac{d}{dt}\left[\int_{CV} \beta \rho \, dU\right] + \int_{CS} \beta \rho \left(\vec{V}_r \cdot \hat{n}\right) dA$$

Example. (Adapted from FM White's Fluid Mechanics)
A baloon is being filled through section I. Area there is A_I, Velocity is V_I, fluid density is ρ_I.
The average density within the baloon is $\rho_b(t)$.
Find an expression for rate of change of fluid system mass within the baloon at this instant.

Solution:

→ For this case, the baloon boundary form the control volume.



→ However, the volume of the control surface increases with time.

We will use RTT.

$$\frac{dB_{system}}{dt} = \frac{d}{dt}\left[\int_{cv} \beta \rho \, dV\right] + \int_{cs} \beta \rho \, \vec{v}_r \cdot \hat{n} \, dA$$

Here B_{system} = Mass of air in the balloon = m_{system}

According to conservation of mass ~~prove~~ law,

$$\frac{d\, m_{system}}{dt} = 0.$$

$$\beta = \frac{dm}{dm} = 1.0.$$

$$\therefore \quad 0 = \frac{d}{dt}\left[\int_{cv} \rho \, dV\right] + \int_{cs} \rho \, (\vec{v}_r \cdot \hat{n}) \, dA.$$

At inlet, $v_r = V_I$ ($\because$ no change in position)

At outlet, $v_r = 0$ ($\because$ air is not escaping)

$$\therefore \quad 0 = \frac{d}{dt}\left[\int_{cv} \rho \, dV\right] + \int_{cs} \rho \times 0 - \rho_1 A_I V_I$$

i.e.
$$\frac{d}{dt}\left[\int_{cv} \rho \, dV\right] = \rho_1 A_I V_I$$

or
$$\frac{d}{dt}\left[\rho_b \frac{4\pi}{3} R^3\right] = \rho_I A_I V_I$$

or
$$\frac{d}{dt}\left(\rho_b R^3\right) = \frac{3}{4\pi} \rho_I A_I V_I$$