Solution of DGLAP evolution equation in Next-to-Leading Order (NNLO) at small-x

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Abstract: The structure functions have been obtained by solving Dokshitzer- Gribov- Lipatov- Alterelli- Parisi (DGLAP) evolution equations in next-next-to-leading order (NNLO) at the small-x limit. Here a Taylor series expansion has been used and then the method of characteristics to solve the evolution equations. Results are compared with the Fermilab experiment E665 and New Muon Collaboration (NMC) data.
Introduction:

**Deep Inelastic Scattering (DIS):**

\[ q = K - K' = - Q^2 \]

\[ P, E_h, M_h \]

\[ \text{hadron} \]

\[ \text{Product (X)} \]

- \( P, E_h, M_h \): Four momentum, Energy, Rest mass of the hadron
- \( q (K-K') \): Four momentum transfer of lepton
- \( X \): Any set of outgoing particles
Structure functions:-

Product (X) can be obtain by knowing -

\[ F_2^S(x, Q^2), F_2^{NS}(x, Q^2), G(x, Q^2) = \text{Structure functions} \]

\[ x = \frac{Q^2}{2p.q} = \text{Fraction of hadrons momentum carried by quark} \]

To know inner structure of hadrons at high energy, we have to study the Structure functions
Evolution Equations

* Dokshitzer-Gribov-Lipatov-Alterelli-Parisi (DGLAP)

* Balitskij-Kuraev-Fadin-Lipatov (BKFL)

* Gribov-Levin-Ryskin (GLR)

* Ciafaloni-Catani-Fiorani-Marchesini (CCFM)
Schematic representation of the applicability of various evolution equations across the $x,Q^2$ plane
Various methods for Solution of Evaluation Equations (DGLAP):

1. Brute-force method,  
2. Use of Laguerre Polynomials  
3. Mellin moment space method  
4. Matrix approach  
5. Taylor’s Expansion Method  
6. Particular Solution Methods  
7. Regge Behavior etc.

These are some numerical as well as analytical solution. But as the evolution equations are Partial Differential Equations (PDE), So getting its unique solution is a difficult task. Hence, we are searching a method for unique solution and we proposed

--------- Method of characteristics
Method of characteristics:---

- The Method of Characteristics is a method that can be used to Solve the initial value problem for general first order partial differential Equation (PDE) –

- The goal of Method of Characteristics is to convert the PDE to ordinary differential equation (ODE) and we get the unique solution.
Method of characteristics:

\[ a(x,t) U_x + b(x,t) U_t + c(x,t) U = 0 \quad (1) \]

With the initial condition \[ U(x,0) = f(x) \quad (2) \]

We are replacing the coordinates \((x, t)\) by new coordinate system \((S, \tau)\) such that –

1) \(S\) changes along a vertical curvy line in the \(x-t\) plane and \(\tau\) is constant.

2) \(\tau\) changes along a horizontal curvy line where \(S\) is const.

For \(t\) evolution, we consider, \(S\) changes along the characteristic curve \([x(S), t(S); 0<S<\infty]\) and \(\tau\) changes along the initial curve \((t=t_0)\). On the other hand, for \(x\)-evolution \(\tau\) changes along the characteristic curve \([x(\tau), t(\tau); 0<\tau<\infty]\) and \(S\) change along the initial curve \((x=x_0)\)
Method of characteristics: --
\textbf{Method of characteristics:---}

Let us consider

\[ \frac{dx}{dS} = a(x, t) \quad \text{and} \quad \frac{dt}{dS} = b(x, t) \]

We know that the relation between Partial Differential and Ordinary Differential is

\[ \frac{dU(x, t)}{dS} = \frac{\partial U(x, t)}{\partial x} \frac{dx}{dS} + \frac{\partial U(x, t)}{\partial t} \frac{dt}{dS} = \frac{dU(S, \tau)}{dS}, \]

\[ \Rightarrow \frac{dU(S, \tau)}{dS} = a(x, t) \frac{\partial U(x, t)}{\partial x} + b(x, t) \frac{\partial U(x, t)}{\partial t} \]

\[ \frac{dU(S, \tau)}{dS} + c(S, \tau)U(S, \tau) = 0 \]
The DGLAP evolution equations for non-singlet structure functions $(in \text{ standard form})$ is

$$\frac{\partial F^{NS}_2(x, Q^2)}{\partial \ln Q^2} = P^{NS}_1(x, Q^2) \otimes F^{NS}_2(x, Q^2)$$

Here $\otimes$ represents the standard Mellin Convolution and the notation is

$$a(x) \otimes b(x) \equiv \int_0^1 \frac{dy}{y} a(y) b\left(\frac{x}{y}\right)$$

The non-singlet kernel is

$$P^{NS}_1(x, Q^2) = \frac{\alpha_s(Q^2)}{2\pi} P^{(0)}_{NS}(x) + \left(\frac{\alpha_s(Q^2)}{2\pi}\right)^2 P^{(1)}_{NS}(x) + \left(\frac{\alpha_s(Q^2)}{2\pi}\right)^3 P^{(2)}_{NS}(x)$$
where $P_{NS}^{(0)}(x)$, $P_{NS}^{(1)}(x)$ and $P_{NS}^{(2)}(x)$ are non-singlet splitting functions in LO, NLO and NNLO respectively. Using $P_{NS}^{(0)}(x)$ and $P_{NS}^{(1)}(x)$ [defined in Ref. W. Furmanski and R. Petronzio, Phys. Lett. B 97, 437 (1980); Nucl. Phys. B 195, 237 (1982)], the DGLAP equation have been solved up to NLO [R. Baishya and J. K. Sarma, Phys. Rev. D 74, 107702 (2006)]. By adding $P_{NS}^{(2)}(x)$ with previous terms we will get the NNLO evolution equation.

Applying all these and simplifying, the DGLAP evolution equations for non-singlet structure function in NNLO can be written as

$$
\frac{\partial F_{2}^{NS}}{\partial t} - \frac{\alpha_s(t)}{2\pi} \left[ \frac{2}{3} \{3 + 4\ln(1-x)\} F_{2}^{NS}(x,t) + I_{1}^{NS}(x,t) \right] - \left( \frac{\alpha_s(t)}{2\pi} \right)^2 I_{2}^{NS}(x,t) - \left( \frac{\alpha_s(t)}{2\pi} \right)^3 I_{3}^{NS}(x,t) = 0
$$
where

\[ I_{1}^{NS}(x, t) = \frac{4}{3} \int_{x}^{1} \frac{d\omega}{1 - \omega} \left[ (1 + \omega^2)F_{2}^{NS}\left(\frac{x}{\omega}, t\right) - 2F_{2}^{NS}(x, t) \right], \]

\[ I_{2}^{NS}(x, t) = (x - 1)F_{2}^{NS}(x, t) \int_{0}^{1} f(\omega) d\omega + \int_{x}^{1} f(\omega)F_{2}^{NS}\left(\frac{x}{\omega}, t\right) d\omega, \]

\[ I_{3}^{NS}(x, t) = \int_{x}^{1} \frac{d\omega}{\omega} \left[ P_{NS}(x)F_{2}^{NS}\left(\frac{x}{\omega}, t\right) \right] \]

with

\[ f(\omega) = C_{F}^{2}[P_{F}(\omega) - P_{A}(\omega)] + \frac{1}{2} C_{F} C_{A} [P_{G} + P_{A}(\omega)] + C_{F} T_{R} N_{f} P_{N_{f}}(\omega) \]

\[ P_{F}(\omega) = -\frac{2(1 + \omega^2)}{(1 - \omega)} \ln(\omega)\ln(1 - \omega) - \left( \frac{3}{1 - \omega} + 2\omega \right) \ln\omega - \frac{1}{2} (1 + \omega)\ln\omega + \frac{40}{3} (1 - \omega) \]
\[ P_G(\omega) = \frac{1+\omega^2}{(1-\omega)} \left( \ln^2(\omega) + \frac{11}{3} \ln(\omega) + \frac{67}{9} - \frac{\pi^2}{3} \right) + 2(1+\omega)\ln\omega + \frac{40}{3}(1-\omega) \]

\[ P_{N_f}(\omega) = \frac{2}{3} \left[ \frac{1+\omega^2}{1-\omega} \left( -\ln\omega - \frac{5}{3} \right) - 2(1-\omega) \right] \]

\[ P_A(\omega) = \frac{2(1+\omega^2)}{(1+\omega)} \left( \frac{1}{1+\omega} \right) \int \frac{dk}{k} \ln\left( \frac{1-k}{k} \right) + 2(1+\omega)\ln(\omega) + 4(1-\omega) \]

\[ P^{(2)}_{NS}(x, t) = n_f \left[ \left\{ L_1 \left( -163.9x^{-1} - 7.208x \right) + 151.49 \right\} (1-x) + 44.51x - 43.12x^2 + 4.82x^3 + L_0 L_1 [-173.1 + 46.18L_0] + 178.04L_0 + 6.892L_0^2 + \frac{40}{27} \left( L_0^4 - 2L_0^3 \right) \right] \]
with $C_A = C_G = 3$, $L_0 = \ln (x)$, $L_1 = \ln(1-x)$.

Now let us introduce the variable $u = 1- \omega$ and since $x < \omega < 1$, so $0 < u < 1-x$, using Taylor’s expansion series, we can rewrite

$$F_2^{NS} \left( \frac{x}{\omega}, t \right) = F_2^{NS}(x, t) + \frac{xu}{1-u} \frac{\partial F_2^{NS}(x, t)}{\partial x}$$

Since $x$ is small in our region of discussion, so higher terms are neglected. Hence, equation of Non-singlet structure in NNLO becomes the form

$$\frac{\partial F_2^{NS}(x, t)}{\partial t} - \frac{\alpha_s}{2\pi} \left[ A_1(x)F_2^{NS}(x, t) + A_2(x) \frac{\partial F_2^{NS}(x, t)}{\partial x} \right]$$

$$- \left( \frac{\alpha_s}{2\pi} \right)^2 \left[ B_1(x)F_2^{NS}(x, t) + B_2(x) \frac{\partial F_2^{NS}(x, t)}{\partial x} \right]$$

$$- \left( \frac{\alpha_s}{2\pi} \right)^3 \left[ C_1(x)F_2^{NS}(x, t) + C_2(x) \frac{\partial F_2^{NS}(x, t)}{\partial x} \right] = 0$$
where

\[ A_1(x) = 2x + x^2 + 4\ln(1 - x) \quad A_2(x) = x - x^3 - 2x\ln(x) \]

\[ B_1(x) = x \int_0^1 f(\omega) d\omega - \int_0^x f(\omega) d\omega + \frac{4}{3} N_f \int_x^1 F_{qq}^S(\omega) d\omega \]

\[ B_2(x) = x \int_x^1 \left[ f(\omega) + \frac{4}{3} N_f F_{qq}^S(\omega) \right] \frac{1 - \omega}{\omega} d\omega \]

\[ C_1(x) = n_f \int_0^{1-x} \frac{d\omega}{1 - \omega} \begin{bmatrix} \ln(\omega) \left[ -163.9(1 - \omega)^{-1} - 7.208(1 - \omega) \right] + 151.49 \\ + 44.51(1 - \omega) - 43.12(1 - \omega)^2 + 4.82(1 - \omega)^3 \\ \ln(\omega)(-173.1 + 46.18\ln(1 - \omega)) + 178.04\ln(1 - \omega) \\ + 6.892\ln^2(1 - \omega) + \frac{40}{27} \left( \ln^4(1 - \omega) - 2\ln^3(1 - \omega) \right) \end{bmatrix} \]
Now we can consider two numerical parameters $T_0$ and $T_1$, such that $T^2(t) = T_0 T(t)$ and $T^3(t) = T_1 T(t)$, where

The NNLO equation is

$$- t \frac{\partial F_{2\text{NS}}(x, t)}{\partial t} + L(x) \frac{\partial F_{2\text{NS}}(x, t)}{\partial x} + M(x) F_{2\text{NS}}(x, t) = 0$$

where

$$L(x) = \frac{3}{2} A_f (A_2 + T_0 B_2 + T_1 C_2)$$

and

$$M(x) = \frac{3}{2} A_f (A_1 + T_0 B_1 + T_1 C_1)$$
To introduce method of characteristics, let us consider two new variables \((S, \tau)\) instead of \((x, t)\), such that

\[
\frac{dt}{dS} = -t \quad \text{and} \quad \frac{dx}{dS} = L(x)
\]

The equation takes the form

\[
\frac{dF^NS_2(S, \tau)}{dS} + M(S, \tau)F^NS_2(S, \tau) = 0
\]

Solution is

\[
F^NS_2(S, \tau) = F^NS_2(\tau) \left( \frac{t}{t_0} \right)^{M(S, \tau)}
\]
Now we have to replace the co-ordinate system \((S, \tau)\) to \((x, t)\) with the input function and will get the \(t\)-evolution of non singlet structure function in the NNLO as

\[
F_{2}^{NS}(x, t) = F_{2}^{NS}(x, t_0) \left( \frac{t}{t_0} \right)^{\frac{3}{2}} A_f (A_1(x) + T_0 B_1(x) + T_1 C_1(x))
\]

Similarly, the \(x\)-evolution of non singlet structure function will be

\[
F_{2}^{NS}(x, t) = F_{2}^{NS}(x_0, t) \exp \left[ - \int_{x_0}^{x} \frac{(A_1(x) + T_0 B_1(x) + T_1 C_1(x))}{(A_2(x) + T_0 B_2(x) + T_1 C_2(x))} \, dx \right]
\]
Results and Discussion:

In this report, we have compared our results of $t$ and $x$-evolutions of non-singlet structure function $F_{2}^{NS}(x, t)$ with E665 experiment data [12] (taken at Fermilab in inelastic muon scattering with an average beam energy of 470 GeV$^2$) and NMC data [13] (in muon-deuteron DIS with incident momentum 90, 120, 200, 280 GeV$^2$). We consider the range $0.01 \leq x \leq 0.0489$ and $1.496 \leq Q^2 \leq 13.391$ GeV$^2$ for E665 data and $0.0045 \leq x \leq 0.14$ and $0.75 \leq Q^2 \leq 20$ GeV$^2$ for NMC data. It is observed that, within these range, for the lowest error the value of $T_0$ and $T_1$ will be $T_0 = 0.05$ and $T_1 = 0.0028$ (Fig. 1).
FIG. 1: $T^2(t)$, $T_0.T(t)$ and $T^3(t)$, $T_1.T(t)$ versus $Q^2 (\text{GeV}^2)$
FIG. 2(a,b): E665 data. Dash-dotted lines are our LO results, dotted lines are our LO + NLO results and solid lines are our LO + NLO + NNLO results. For clarity, data are scaled up by $+0.5i$ ($i=0, 1, 2, 3$) starting from bottom of all graphs in each figure.
FIG. 3(a,b): NMC data. Desh-dotted lines are our LO results, dotted lines are our LO + NLO results and solid lines are our LO + NLO + NNLO results. For clarity, data are scaled up by +0.3i (in Fig. 3(a)) and +0.2i (in Fig. 3(b)) (with $i=0, 1, 2, 3$) starting from bottom of all graphs in each figure.
Conclusion:

The contribution of NNLO is found to be high at the lower-x and higher-$Q^2$. It is observed that, our results are compatible with experimental values. Moreover, it is observed from the figures that LO, NLO and NNLO total contributions are better in higher-$Q^2$ and smaller-x region. This is so because our theory is better in that region (DGLAP).
References:


M. Stramann and W. Vogelsang, Phys. Rev. D 64, 114007 (2001)
(1997)
Thanks