Maximal Independent Set

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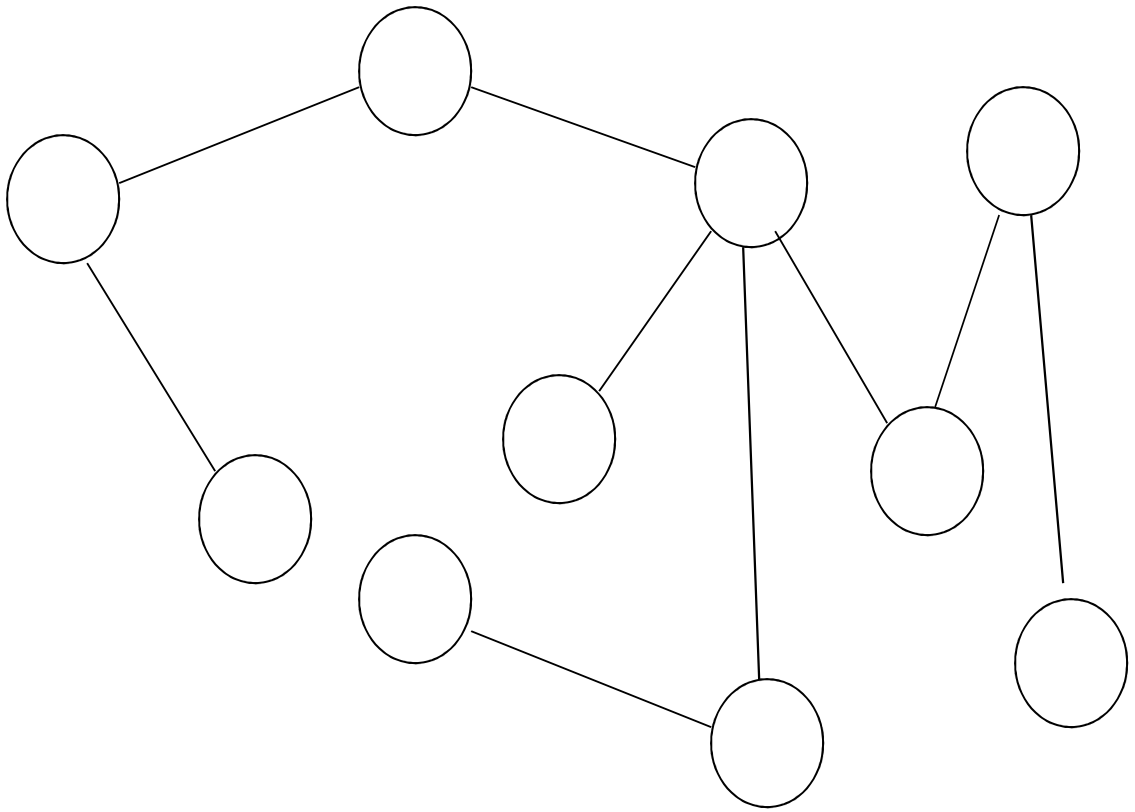
What is a MIS?

MIS

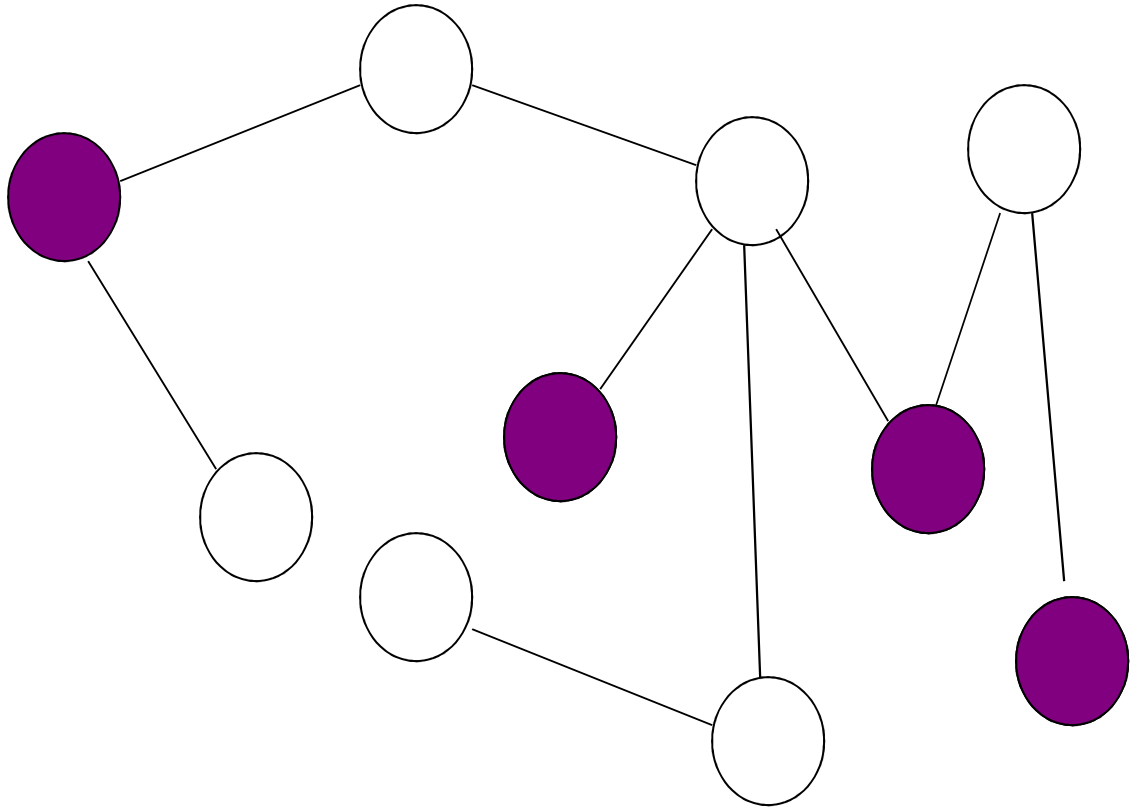
An independent set (IS) of an undirected graph is a subset U of nodes such that no two nodes in U are adjacent. An IS is maximal if no node can be added to U without violating IS (called MIS). A maximum IS (called MaxIS) is one of maximum cardinality.

Known from „classic TCS“: applications?
Backbone, parallelism, etc.
Also building block to compute matchings and coloring!
Complexities?

MIS and MaxIS?

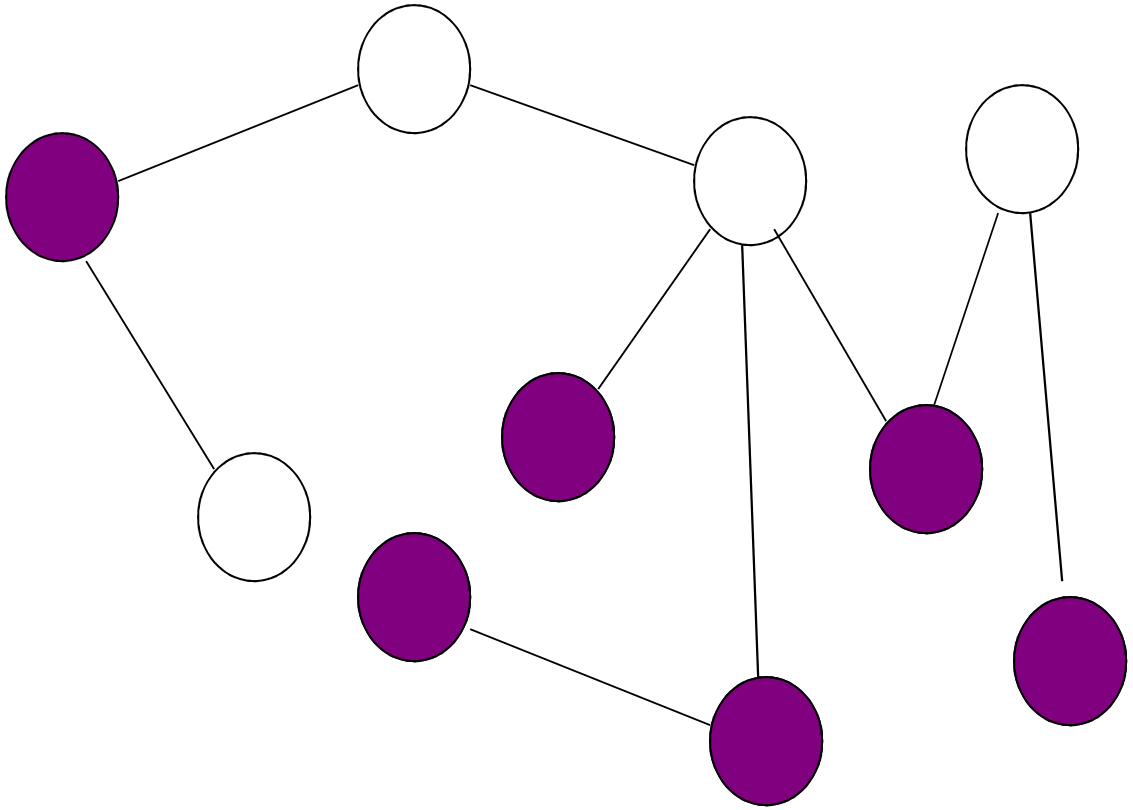


Nothing, IS, MIS, MaxIS?



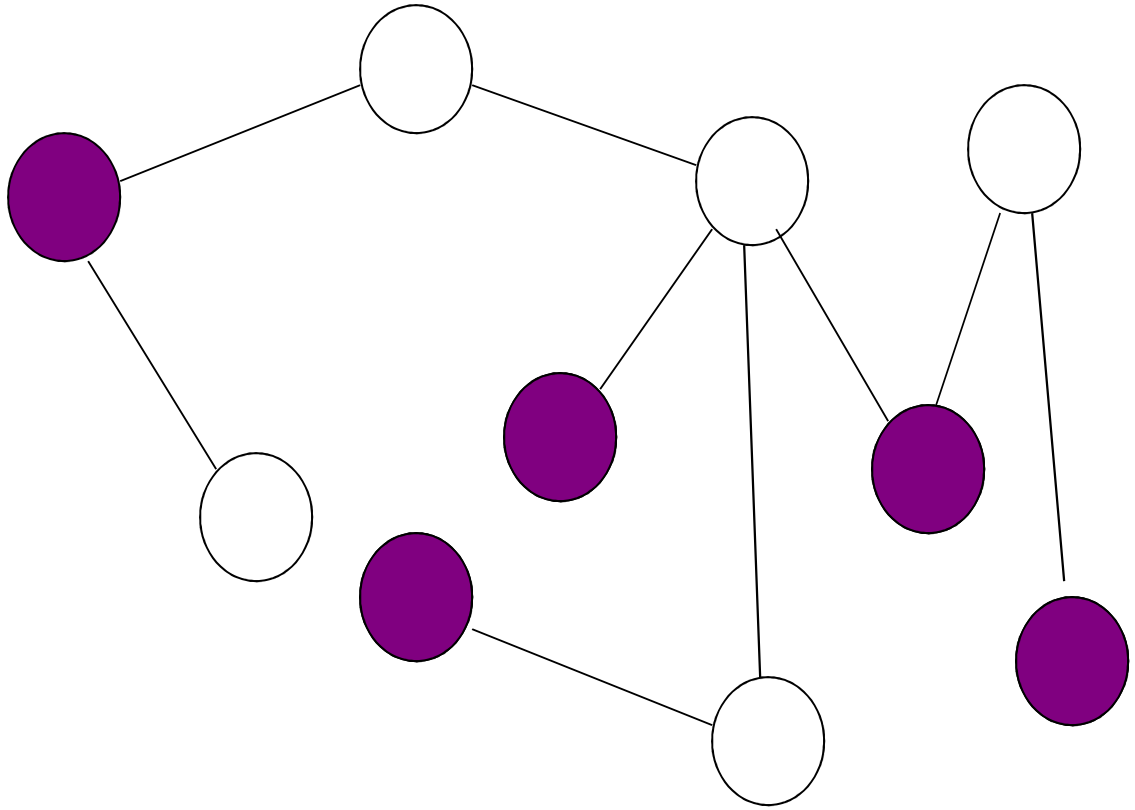
IS but not MIS.

Nothing, IS, MIS, MaxIS?



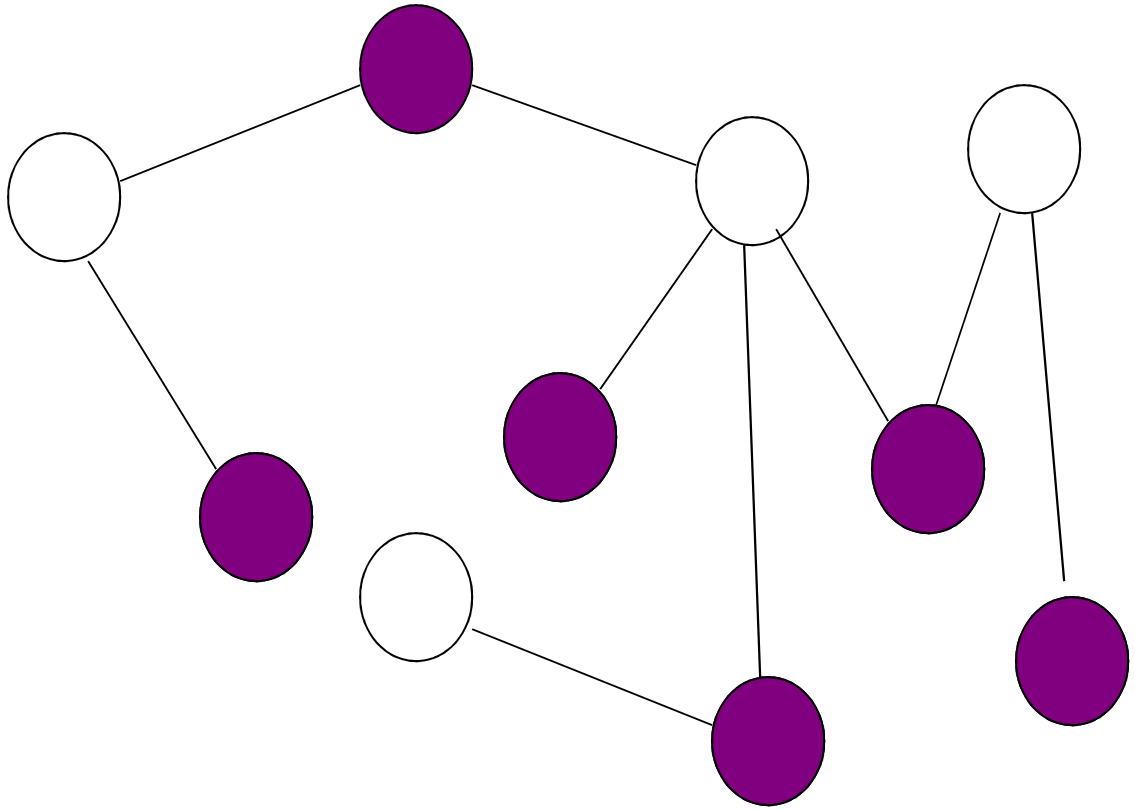
Nothing.

Nothing, IS, MIS, MaxIS?



MIS.

Nothing, IS, MIS, MaxIS?



MaxIS.

Complexities?

MaxIS is NP-hard!

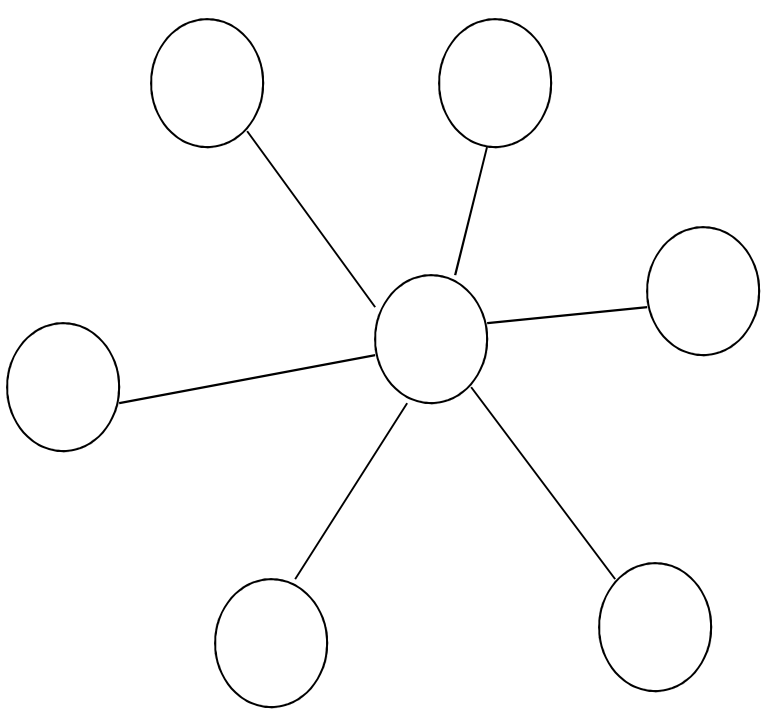
So let‘s concentrate on MIS...

How much worse can MIS be than MaxIS?

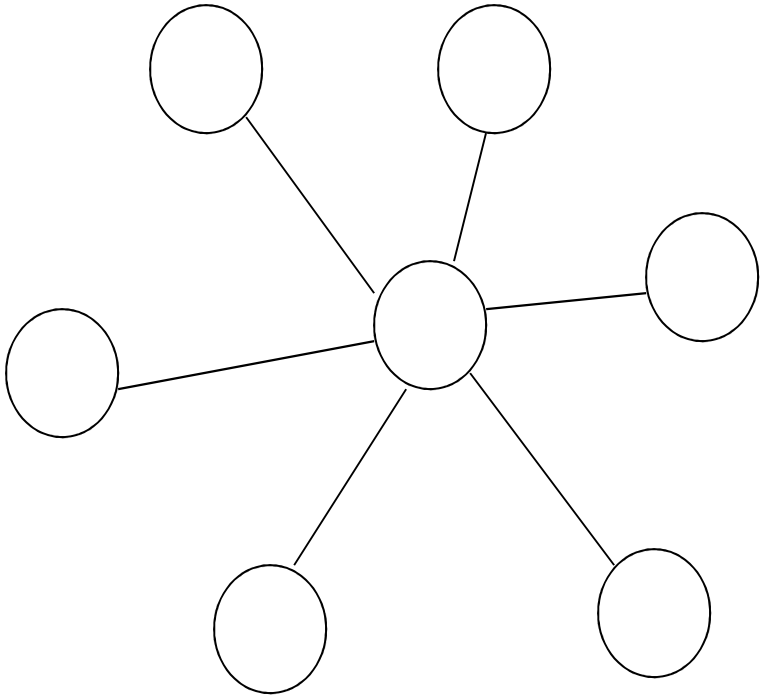
MIS vs MaxIS

How much worse can MIS be than MaxIS?

minimal MIS?



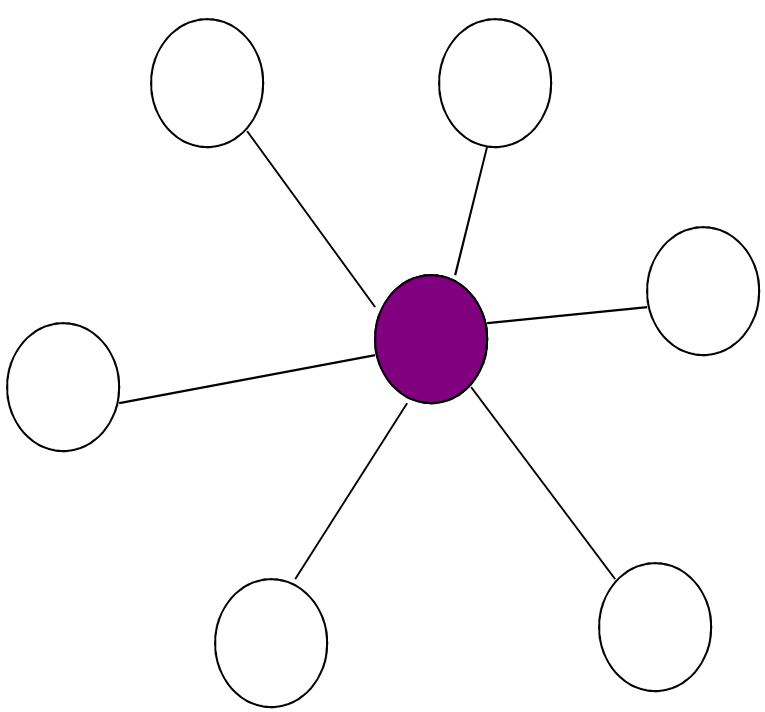
maxIS?



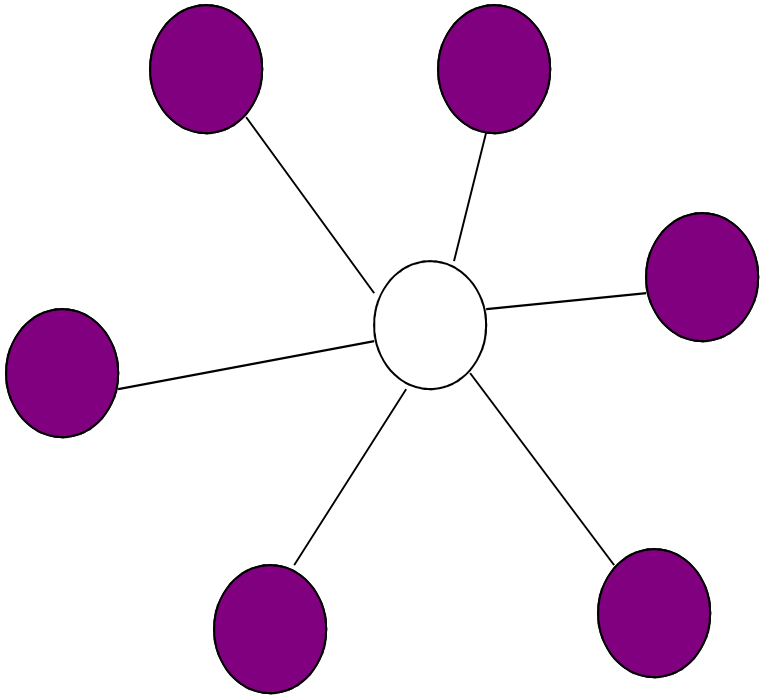
MIS vs MaxIS

How much worse can MIS be than Max-IS?

minimal MIS?



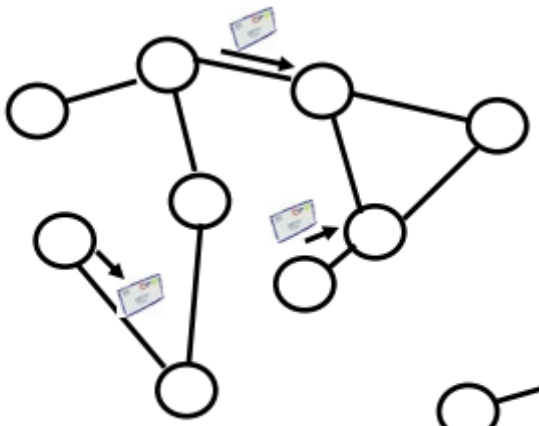
Maximum IS?



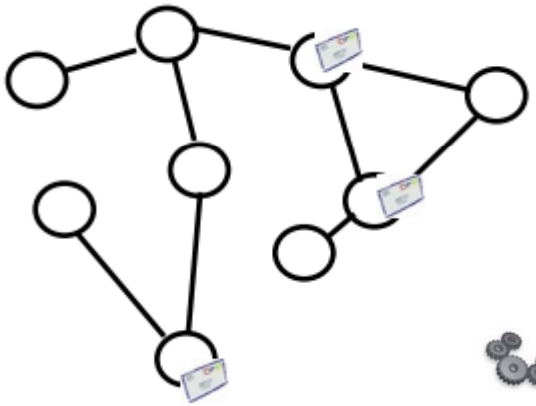
How to compute a MIS in a distributed manner?!

Recall: Local Algorithm

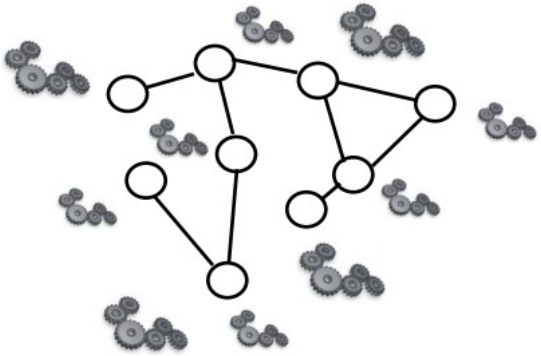
Send...



... receive...



... compute.



Slow MIS

assume node IDs
Each node v:
1. If all neighbors with larger IDs have decided not to join MIS then:
   v decides to join MIS

Analysis?

Analysis

Time Complexity?

Not faster than sequential algorithm!
Worst-case example?
E.g., sorted line: O(n) time.

Local Computations?

Fast! ☺

Message Complexity?

For example in clique: $O(n^2)$
(O(m) in general: each node needs to inform all neighbors when deciding.)

MIS and Colorings

Independent sets and colorings are related: how?

Each color in a valid coloring constitutes an independent set (but not necessarily a MIS, and we must decide for which color to go *beforehand*, e.g., color 0!).

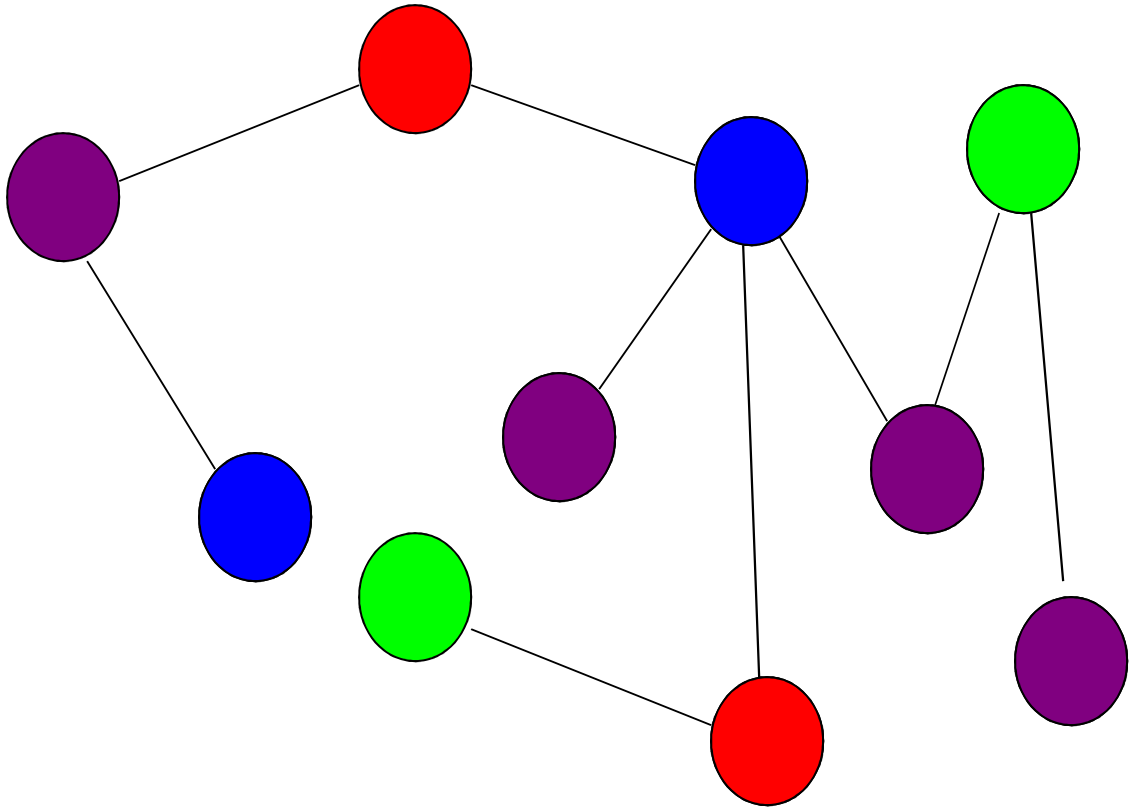
How to compute MIS from coloring?

Choose all nodes of first color. Then for any additional color, add in parallel as many nodes as possible! (Exploit additional independent sets from coloring!)

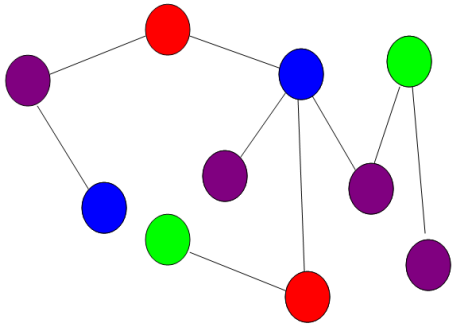
Why, and implications?

Coloring vs MIS

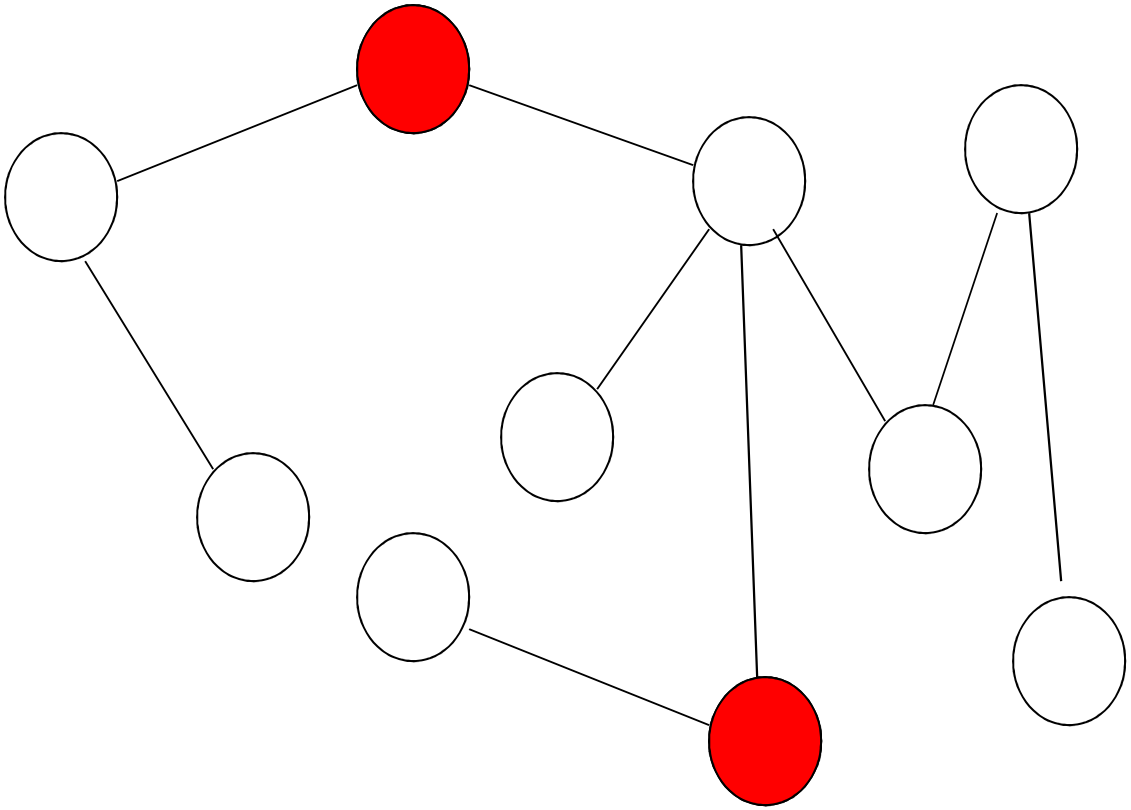
Valid coloring:



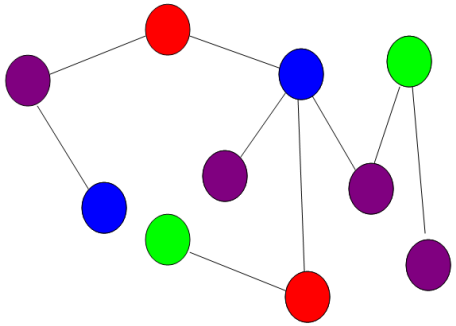
Coloring vs MIS



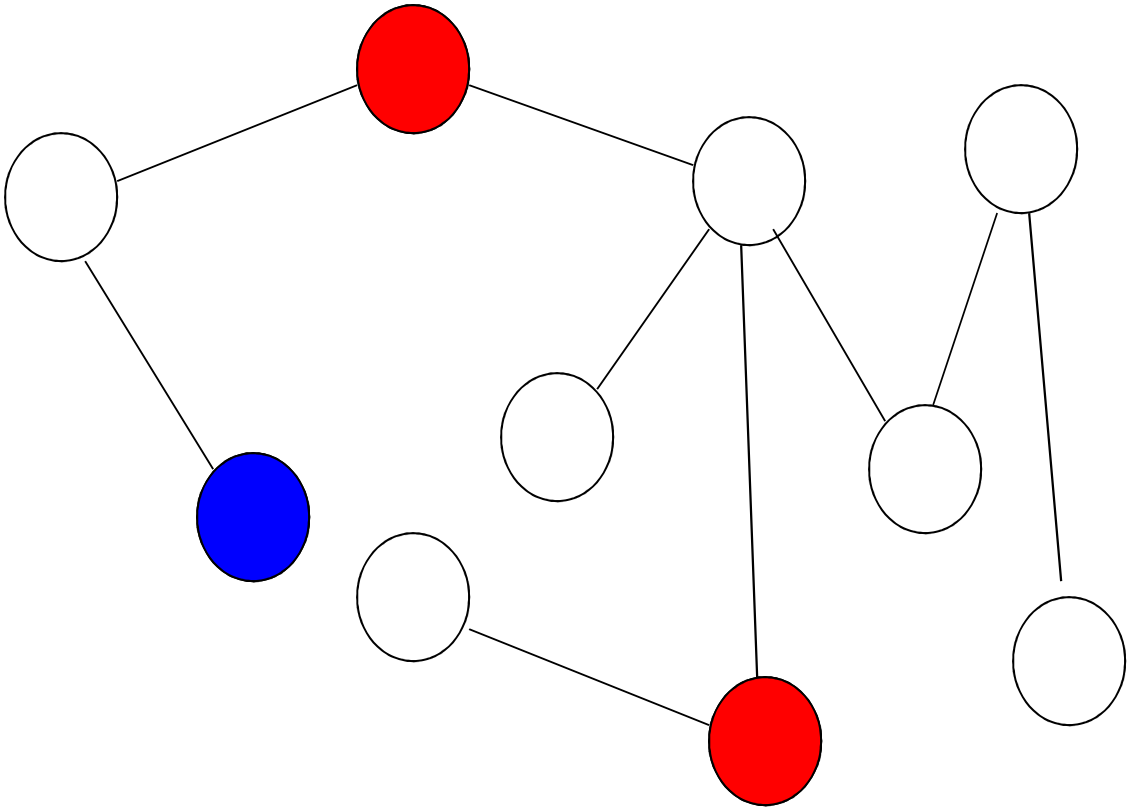
Independent set:



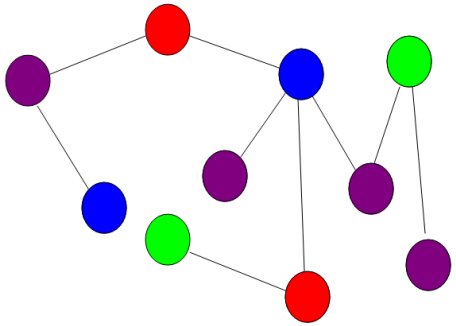
Coloring vs MIS



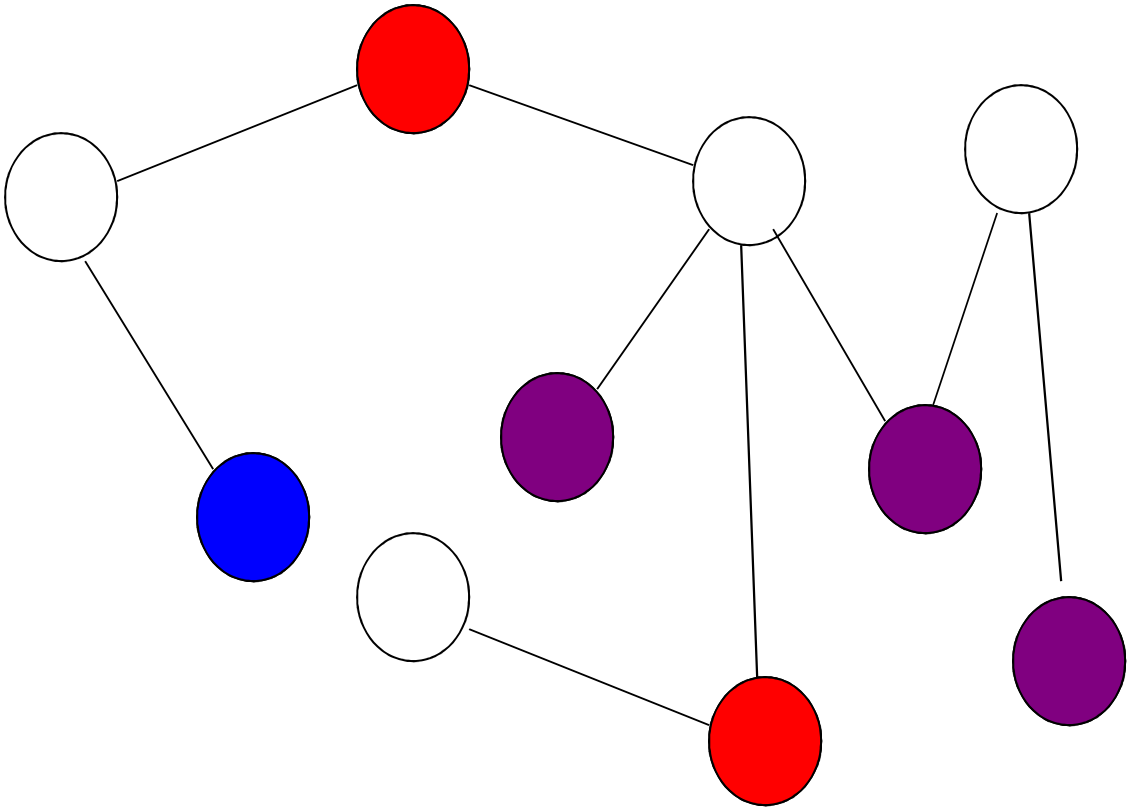
Add all possible blue:



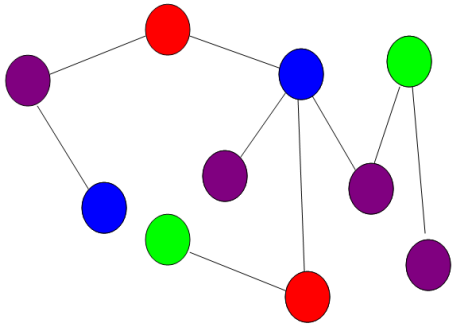
Coloring vs MIS



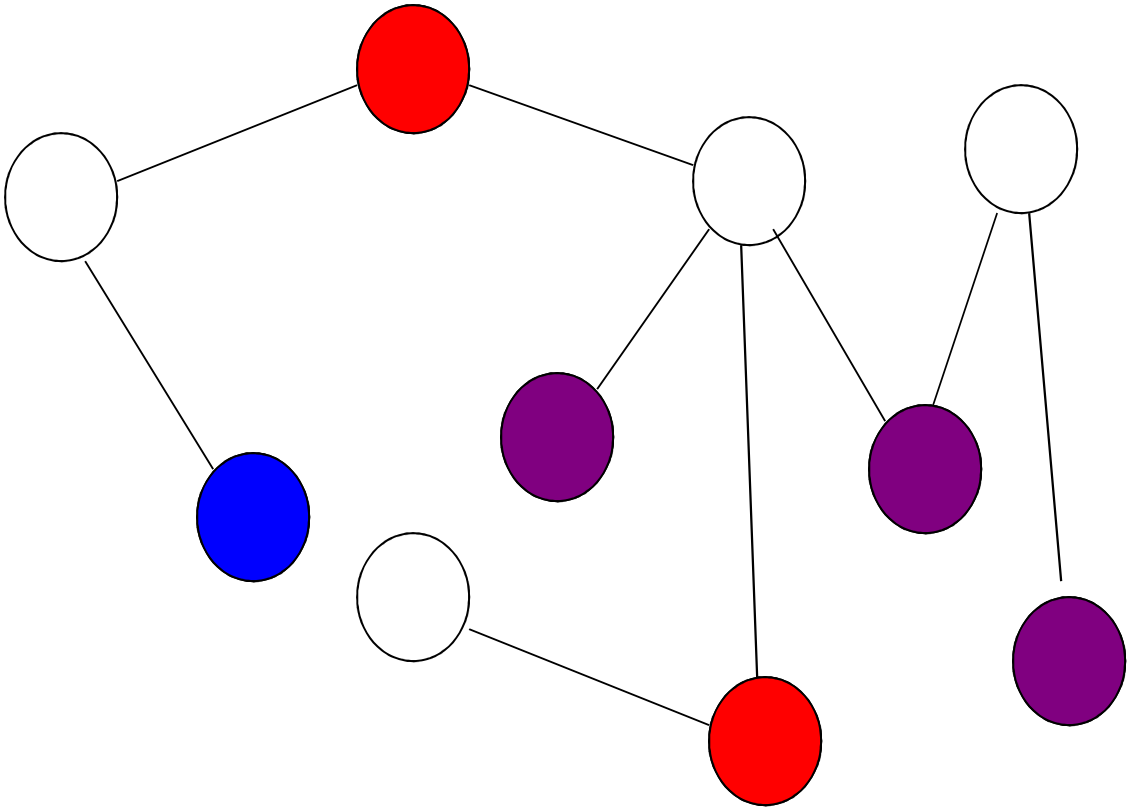
Add all possible violet:



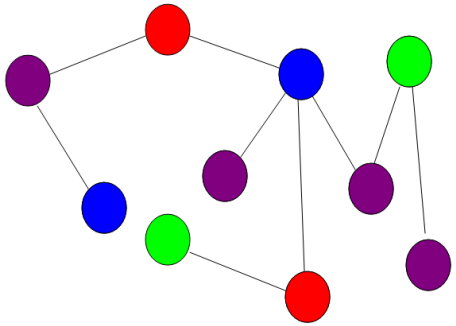
Coloring vs MIS



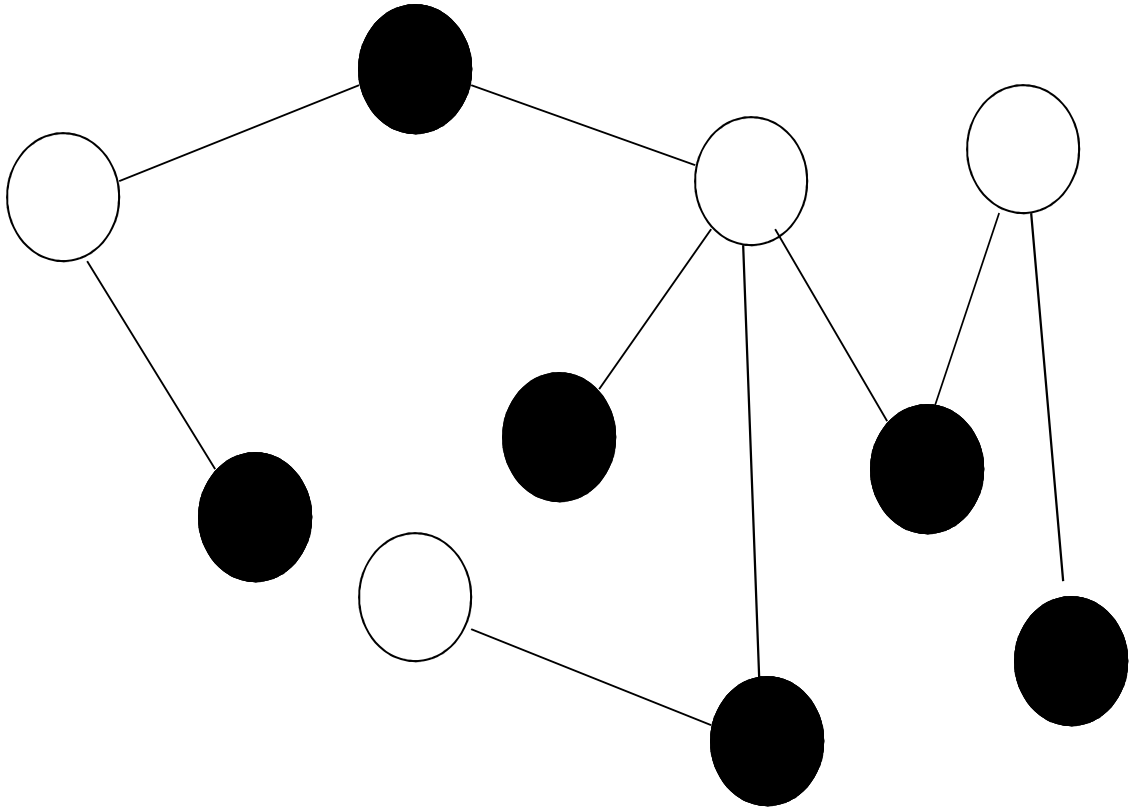
Add all possible green:



Coloring vs MIS



That‘s all: MIS!



Analysis of algorithm?

Analysis

Why does algorithm work?

Same color: all nodes independent, can add them in parallel without conflict (not adding two conflicting nodes concurrently).

Runtime?

Lemma

Given a coloring algorithm with runtime T that needs C colors, we can construct a MIS in time C+T.

Discussion

What does it imply for MIS on trees?

We can color trees in log* time and with 3 colors, so:

MIS on Trees

There is a deterministic MIS on trees that runs in distributed time O(log* n).

Better MIS Algorithms

Any ideas?

Takeaway

If you can‘t find fast deterministic algorithms, try randomization!

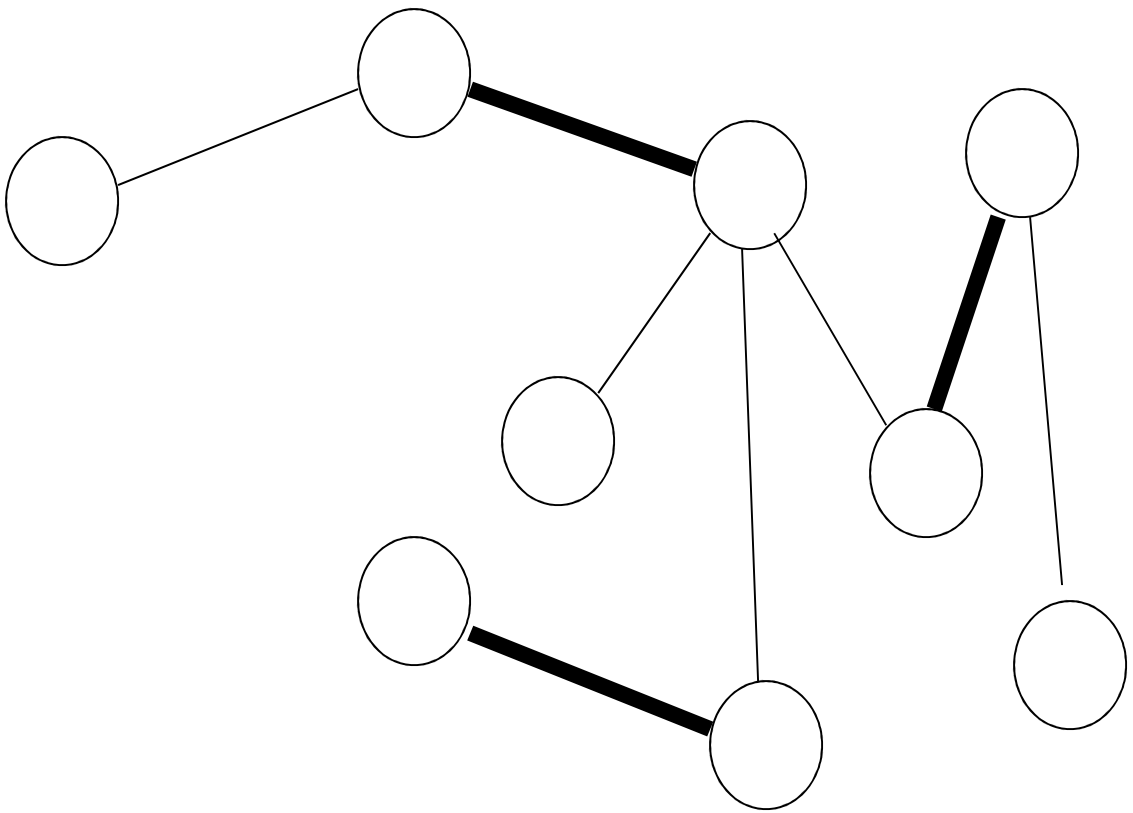
Ideas for randomized algorithms?

Excursion: Matchings

Matching

A matching is a subset M of edges E such that no two edges in M are adjacent.
A maximal matching cannot be augmented.
A maximum matching is the best possible.
A perfect matching includes all nodes.

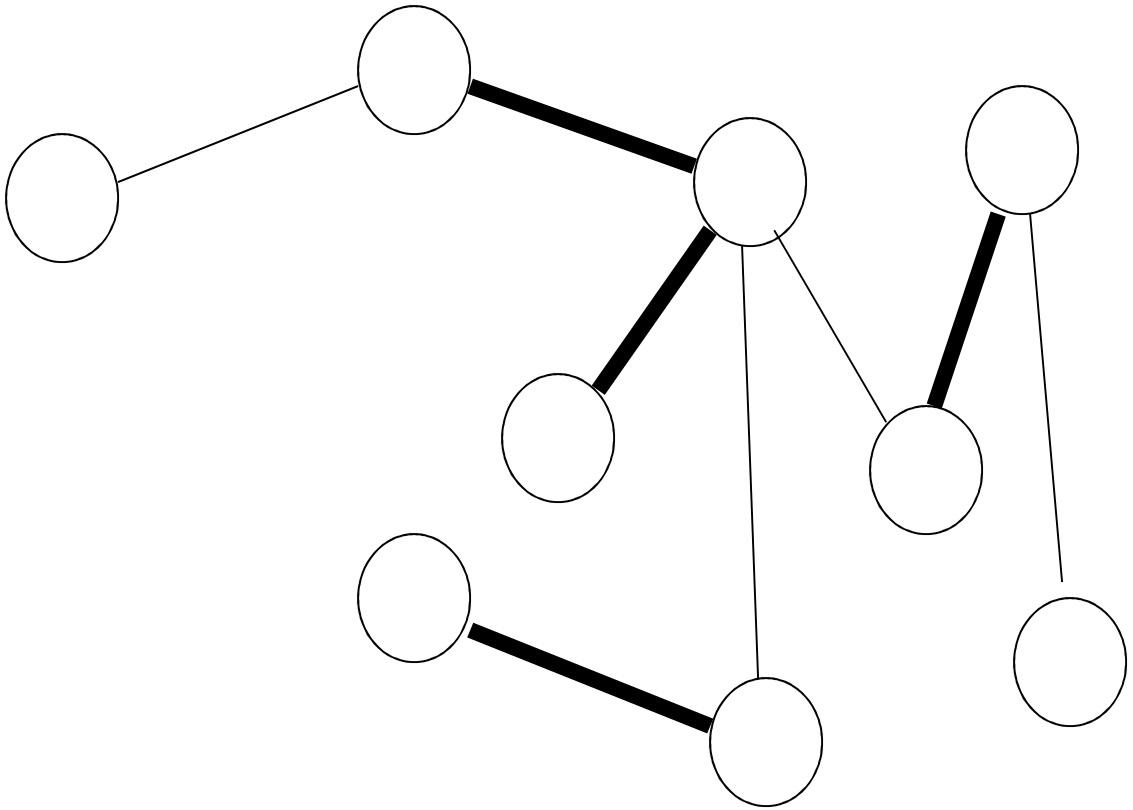
Excursion: Matchings



Matching? Maximal? Maximum? Perfect?

Maximal.

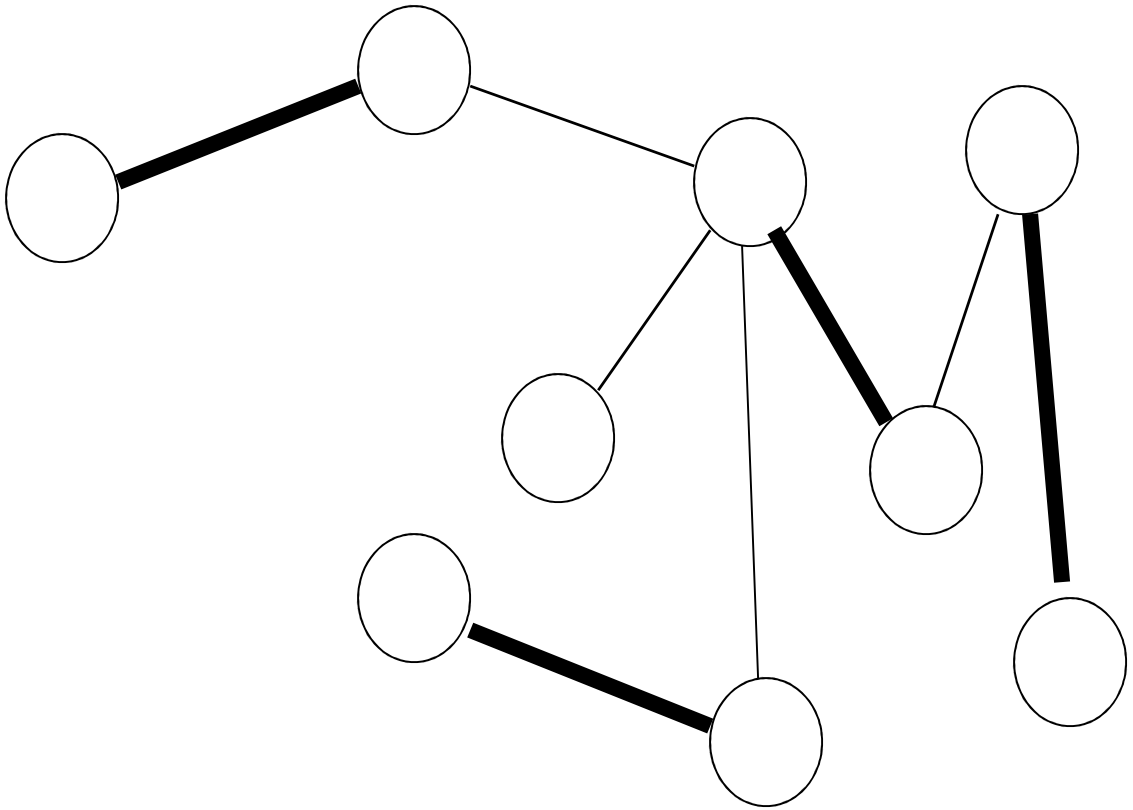
Excursion: Matchings



Matching? Maximal? Maximum? Perfect?

Nothing.

Excursion: Matchings



Matching? Maximal? Maximum? Perfect?

Maximum but not perfect.

Discussion: Matching

Matching

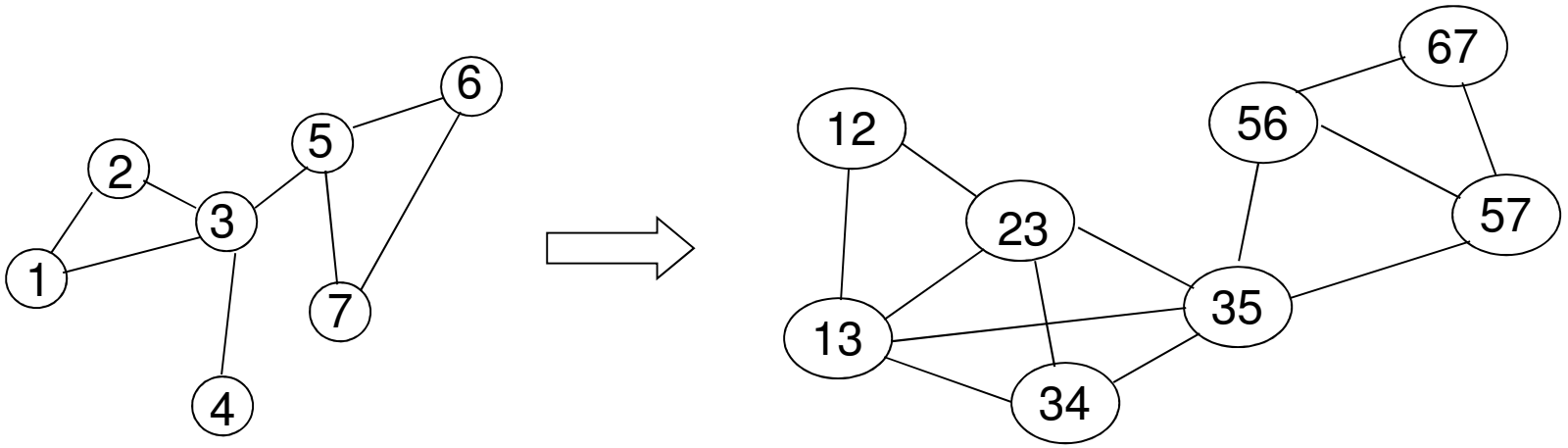
A matching is a subset M of edges E such that no two edges in M are adjacent.
A maximal matching cannot be augmented.
A maximum matching is the best possible.
A perfect matching includes all nodes.

How to compute with an IS algorithm?

Discussion: Matching

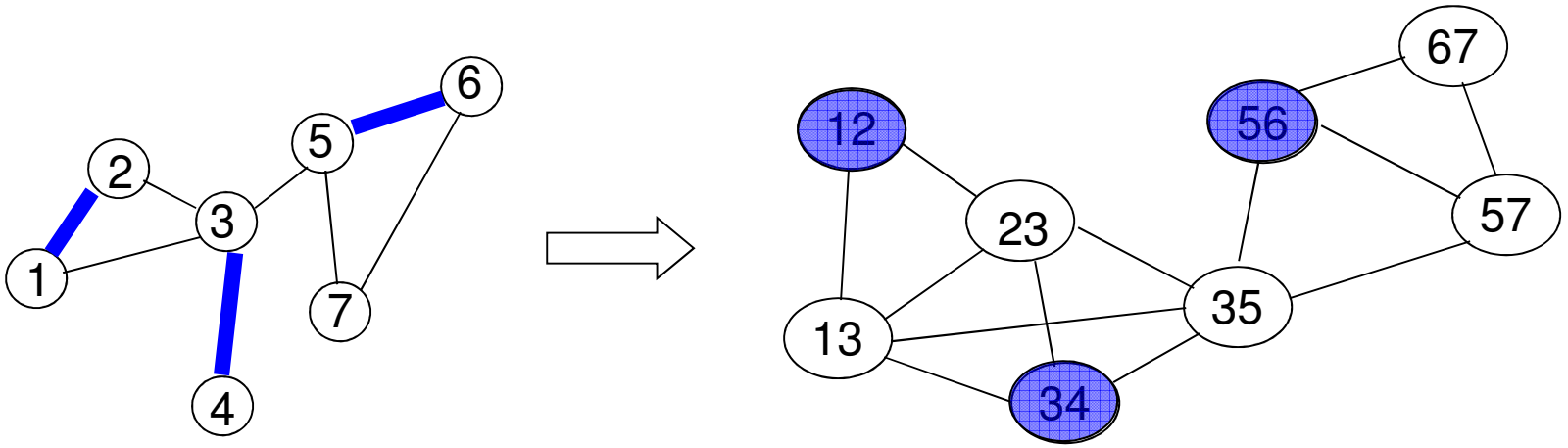
An IS algorithm is a matching algorithm! How?

For each edge in original graph make vertex, connect vertices if their edges are adjacent.



Discussion: Matching

MIS = maximal matching: matching does not have adjacent edges!



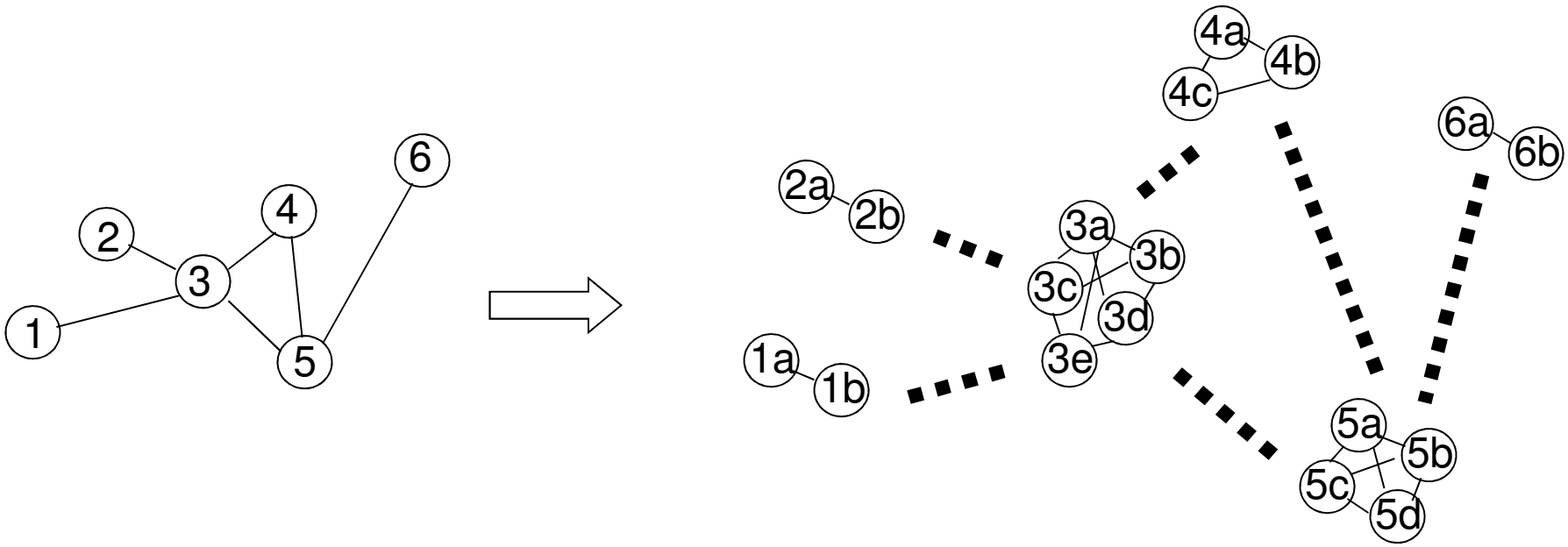
Discussion: Graph Coloring

How to use a MIS algorithm for graph coloring?

How to use a MIS algorithm for graph coloring?

Clone each node v, d(v)+1 many times. Connect clones completely and edges from i-th clone to i-th clone. Then?

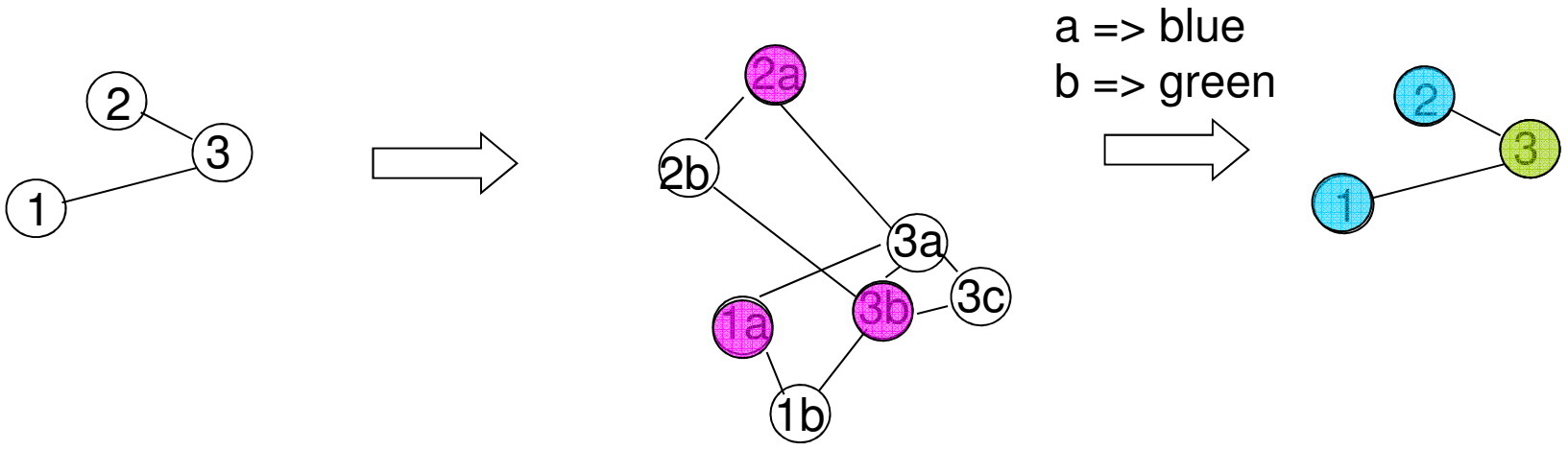
Run MIS: if i-th copy is in MIS, node gets color i.



Discussion: Graph Coloring

Example:

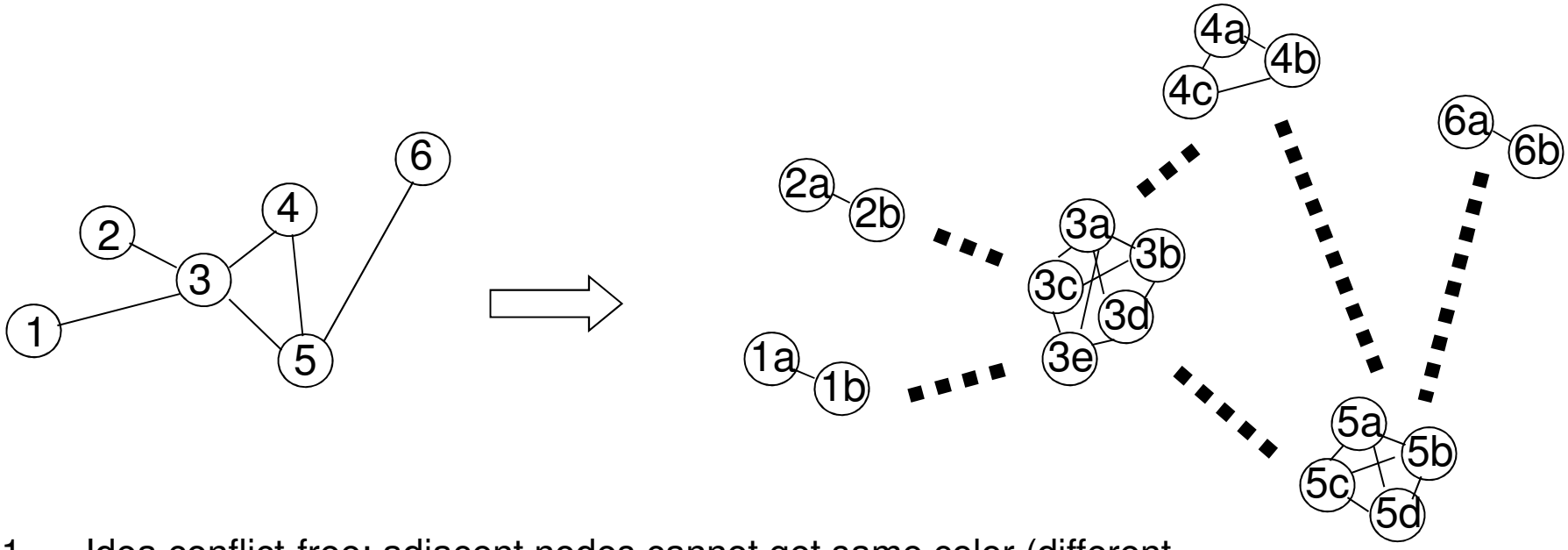
How to use a MIS algorithm for graph coloring?



MIS Coloring

Discussion: Graph Coloring

Why does it work?



1. Idea conflict-free: adjacent nodes cannot get same color (different index in MIS, otherwise adjacent!), and each node has at most one clone in IS, so valid.
2. Idea colored: each node gets color, i.e., each node has a clone in IS: there are only d(v) neighbor clusters, but our cluster has d(v)+1 nodes...

Discussion: Dominating Set

Dominating Set

A subset D of nodes such that each node either is in the dominating set itself, or one of ist neighbors is (or both).

How to compute a dominating set?

Literature for further reading:

- Peleg‘s book

End of lecture