INTRODUCTION TO VIBRATION AND STABILITY ANALYSIS OF MECHANICAL SYSTEMS

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Abstract
In this lecture note the vibration of linear and nonlinear dynamical systems has been briefly discussed. Both inertia and energy based approaches have been introduced to derive the equation of motion. With the help of simple numerical examples, responses of linear and nonlinear systems, their stability and bifurcations have been studied.

1. Introduction

Any motion that repeats itself after an interval of time is called vibration or oscillation. The swinging of a pendulum (Fig.1) and the motion of a plucked string are typical examples of vibration. The theory of vibration deals with the study of oscillatory motion of bodies and forces associated with them.

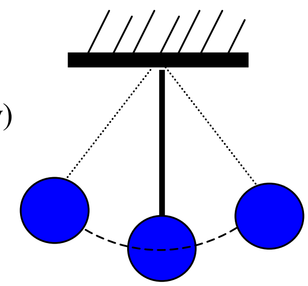


Fig. 1: Swinging of a Pendulum

Elementary Parts of Vibrating system

- A means of storing potential energy (Spring or elasticity)
- A means of storing kinetic energy (Mass or inertia)
- A means by which energy is gradually lost (damper)

The forces acting on the systems are

- Disturbing forces
- Restoring force
- Inertia force
- Damping force

Degree of Freedom: The minimum number of independent coordinates required to determine completely the position of all parts of a system at any instant of time defines the degree of freedom of the system.
System with a finite number of degrees of freedom are called discrete or lumped parameter system, and those with an infinite number of degrees of freedom are called continuous or distributed systems.

Classification of Vibration:

- Free and forced
- Damped and undamped
- Linear and nonlinear
- Deterministic and Random

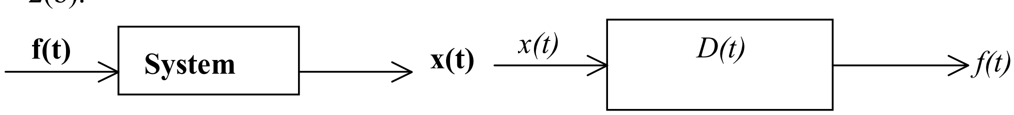
Free vibration: If a system after initial disturbance is left to vibrate on its own, the ensuing vibration is called free vibration.
Forced Vibration: If the system is subjected to an external force (often a repeating type of force) the resulting vibration is known as forced vibration

Damped and undamped: If damping is present, then the resulting vibration is damped vibration and when damping is absent it is undamped vibration. The damped vibration can again be classified as under-damped, critically-damped and over-damped system depending on the damping ratio of the system.

Linear vibration: If all the basic components of a vibratory system – the spring the mass and the damper behave linearly, the resulting vibration is known as linear vibration. Principle of superposition is valid in this case.
Nonlinear Vibration: If one or more basic components of a vibratory system are not linear then the system is nonlinear.
Depending on excitation:
Deterministic: If the value or magnitude of the excitation (force or motion) acting on a vibratory system is known at any given time, the excitation is called deterministic. The resulting vibration is known as deterministic vibration.
Random Vibration: In the cases where the value of the excitation at any given time can not be predicted. Ex. Wind velocity, road roughness and ground motion during earth quake.
Coordinate:
In Newtonian mechanics motions are measured relative to an inertial reference frame, i.e, a reference frame at rest or moving uniformly relatively to an average position of " fixed stars" and displacement, velocity and acceleration are absolute values.
Generalized coordinate: These are a set of independent coordinates same in number as that of the vibrating system. For example, the motion of a double pendulum in planar motion can be represented completely either by θ_1, θ_2 the rotation of the first and second link respectively or by x_1, y_1, x_2, y_2 the Cartesian coordinates of first and second links. While in the later case 4 coordinates are required to represent completely the system, in the former case only 2 coordinates are required for the same. Hence, in this case θ_1, θ_2 is the generalized co-ordinate while x_1, y_1, x_2, y_2 are not the generalized one. One may note that these four coordinates are not independent and can be reduced to two by the use of length constraint.

2. Linear and Nonlinear systems

A system is said to be linear or nonlinear depending on the force response characteristic of the system. The block diagram relating to output and input can be represented as shown in Fig 2(a) and mathematically represented as shown in Fig. 2(b).



(a) (b)

Fig. 2 Block diagram showing the force-response and mathematically relating the input to the output through the operator *D*.

A linear system may be of first or second order depending on the presence of the basic elements. Atypical first order system with linear spring and viscous damping is shown in Fig 3(a) and that of a second order system is shown in Fig.3 (b) as they can be represented by $c\dot{x} + kx = F(t)$ and $m\ddot{x} + c\dot{x} + kx = F(t)$ respectively.

As shown in Fig. 2(b), a system can be represented by using a operator D such that $Dx(t) = f(t)$, where *D* is the differential operator, x(t) is the response and f(t) is the excitation input.
A system $Dx(t) = f(t)$ is said to be linear of it satisfies the following two condition.

1. The response to $\alpha f(t)$ is $\alpha x(t)$, where α is a const.
2. The response to $f_1(t)+f_2(t)$ is $x_1(t)+x_2(t)$ where the $x_1(t)$ is the response to $f_1(t)$ and $x_2(t)$ response $f_2(t)$

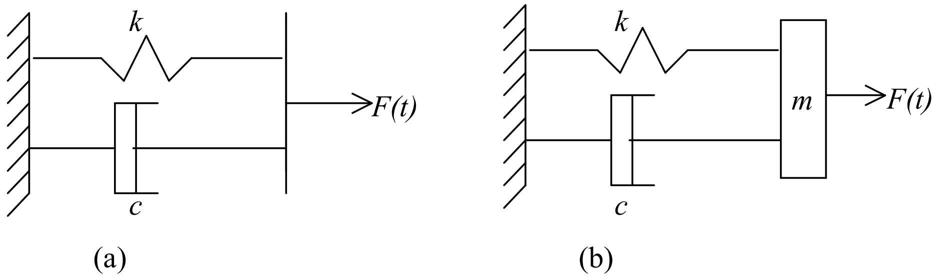


Fig 3(a) First order system (b) second order system

In case $D(\alpha\times(t))=\alpha D\times(t)$, the operator D and hence the system is said to posses homogeneity property and when $D\left[x_1(t)+x_2(t)\right]=Dx_1(t)+Dx_2(t)$ the system is said to posses *additive* property. If an operator D does not possess the homogeneity and additivity property the system is said to be nonlinear.

Example 1: Check whether system given by the following is linear or nonlinear

$$Dx(t)=a_0(t)\frac{d^2\times(t)}{dt^2}+a_1(t)\frac{dx(t)}{dt}+a_2(t)\left[1-\in x^2(t)\right]\times(t) \text{ where, } \in \text{ is a const ant}$$

Solution: check the homogeneity

$$D\left[\alpha x(t)\right]=a_0(t)\alpha\frac{d^2x(t)}{dt^2}+a_1(t)\alpha\frac{dx(t)}{dt}+a_2(t)\alpha\left[1-\in\alpha^2x^2(t)\right]x(t)\ \neq\alpha D(t)$$

Hence homogeneity condition is not satisfied

Similarly substituting $x(t)=x_1(t)+x_2(t)$

$D\left[x_1(t)+x_2(t)\right]\neq Dx_1t)+Dx_2(t).$ which does not satisfy additive property also. Hence the system is a nonlinear system. It may be noted that, the term containing $\in$ causes the nonlinearity of the system. If $\in=0$, the equation becomes linear by satisfying homogeneity and additive properties.

Hence it may be observed that

1) A system is linear if the function $x(t)$ and its derivatives appear to the first (or zero) power only; otherwise the system is nonlinear.
2) A system is linear if a_0, a_1 and a_2 depend as time alone, or they are constant.

3.Equivalent system

The complex vibrating system can be reduced to simpler one by using the concept of equivalent system. The equivalent spring stiffness can be obtained by equating the potential energy of the actual system with that of the equivalent system; equivalent mass or inertia can be obtained by equating the kinetic energy. Similarly equivalent damping can be obtained by equating the energy dissipated per cycle between the actual and the equivalent system.

Example 2: Determine the Equivalent stiffness of the crane shown in fig. 4(a) in the vertical direction.

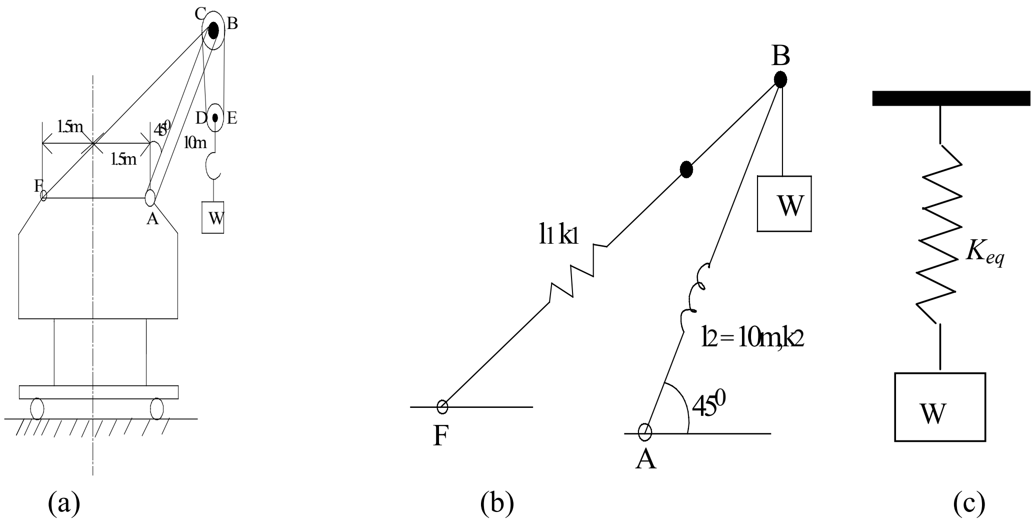


Fig 4. (a) A crane system, (b) simplified model,(c) equivalent spring mass system

The P.E (U) stored in the spring k_1 and k_2 can be expressed

$$U = \frac{1}{2}k_1(x\cos 45^0)^2 + \frac{1}{2}k_2\left[x\cos(90^0-\theta)^2\right]$$

$$k_1 = \frac{A_1E_1}{l_1} = \frac{(100\times10^{-6})(207\times10^9)}{12.3055} = 1.6822\times10^6\ N/m$$

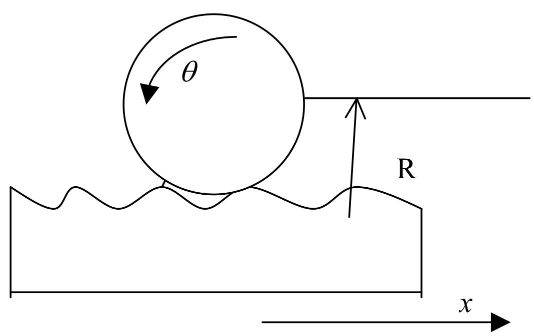
$$k_2 = \frac{A_2E_2}{l_2} = 5.1750\times10^7\ N/m$$

$$Ueq = \frac{1}{2}k_{eq}x^2$$

$$U = U_{eq} \Rightarrow k_{eq} = 26.4304\times10^6\ N/m$$

Equivalent mass

Example 3: Determine the equivalent mass / moment of inertia of the gear-pinion system shown in Fig 5.



$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}J\dot{\theta}^2$$

$$Teq = \frac{1}{2}m_e\dot{x}_{es}^{\ 2}$$

$$\dot{x}_e = \dot{x}$$

$$\dot{\theta} = \dot{x}/R$$

Fig. 5 Gear and pinion system

Equating the kinetic energy of the actual system with that of the equivalent system consisting of mass m_e having translational velocity $\dot{x}$ the equivalent mass is obtained as follow.

$$\frac{1}{2}m_e\dot{x}^2 = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}J\left(\frac{\dot{x}}{R}\right)^2 \Rightarrow m_e = m + \frac{J}{R^2}$$

Similarly one may think of an equivalent system consisting of a gear of moment of inertia J_e and rotating with angular velocity $\dot{\theta}$ and get J_e.

$$\frac{1}{2}J_e\dot{\theta}^2 = \frac{1}{2}m(\dot{\theta}R)^2 + \frac{1}{2}J\dot{\theta}^2 \Rightarrow J_e = J + mR^2$$

Example 4. Table 1 shows the equivalent viscous damping for other types of damped system.

Table 1:

Types of damping	Damping Force	maximumDissipation energy	Equivalent Viscous damping
Viscous damping	$c\dot{x}$	$\pi c\omega X^2$	c
Coulomb damping	$\mu N\,\mathrm{sgn}(\dot{x}), N = mg$	$4\mu NX$	$4\mu N/\pi\omega X$
Structural, solid or Hysteretic damping	$\pi\kappa\beta_h\,\mathrm{sgn}(\dot{x})\lvert x\rvert$	$\pi\kappa\beta_h\int\mathrm{sgn}(\dot{x})\lvert x\rvert\dot{x}dt$	Exercise problem

4. Steps for Vibration Analysis

- Convert the physical system to simplified mathematical model
- Determine the equation of motion l of the system
- Solve the equation of motion to obtain the response
- Interpretation of the result for the physical system.

To convert the physical system into simpler models one may use the concept of equivalent system. To determine the equation of motion basically one may use either the vector approach with the Newtonian approach or d'Alembert principle based on free body diagram or one may go for scalar approach using the energy concept. In scalar approach one may use Lagrange method, which is a differential procedure or extended Hamilton's principle based on integral procedure. Different methods/laws/principle used to determine the equation of motion of the vibrating systems are summarized below.

5. Derivation of Equation of motion

5.1 Newton's second law A practical acted upon by a force moves so that the force vector is equal to the time rate of change of the linear momentum vector.

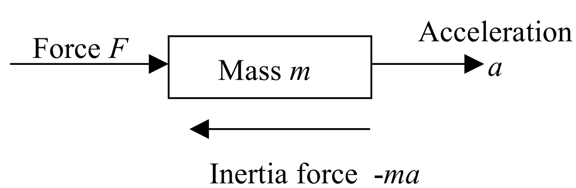


Fig. 6 Application of Newton's second law

Taking, v_i-initial velocity, v_f-final velocity, and t time, according to Newton's 2nd Law

$$F = m\left(\frac{v_f - v_i}{t}\right) = m\,a$$

Example 5 Use Newton's 2nd law to derive equation of motion of a simple pendulum

Solution:

Applying Newton' second law using the coordinate system shown in the free body diagram (Fig 7(b))

$$F = (-T + mg\cos\theta)\hat{I} - mg\sin\theta\ \hat{J}$$
$$= m(l\ddot{\theta}\,\hat{j} - l\dot{\theta}^2\hat{I})$$

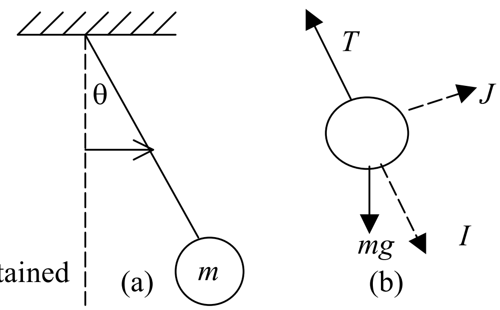


Fig. 7. (a) Simple pendulum
(b) Free body diagram

Now the expression for tension can be obtained by equating the I^{th} component

$T = mg\cos\theta + ml\dot{\theta}^2 = m(L\dot{\theta}^2 + g\cos\theta)$

and the equation of motion can be obtained from the J th component. Hence the equation of motion is

$ml\ddot{\theta} + mg\sin\theta = 0$ or $\ddot{\theta} + \frac{g}{l}\sin\theta = 0$

Also one may use momentum equation i.e., the moment of a force about fixed point is equal to the time rate of change of the angular momentum about that point to obtain the above equation of motion. The above equation is linear only for small value of θ.

5.2 Work energy principle

The work performed in moving the particle from position $\vec{r}_1$ to $\vec{r}_2$ is equal to the change in kinetic energy.

$$\int_{\vec{r}_1}^{\vec{r}_2} \vec{F} \bullet d\vec{r} = \int_{r_1}^{r_2} d\left(\frac{1}{2}m\dot{\vec{r}}\dot{\vec{r}}\right) = \frac{1}{2}m\dot{\vec{r}}_2 \circ \dot{\vec{r}}_2 - \frac{1}{2}m\dot{\vec{r}}_1 \circ \dot{\vec{r}}_1 = T_2 - T_1$$

It can be shown that

- Force for which the work performed in moving a particle over a closed path is zero (considering all possible paths) are said to be *conservative force*.
- Work performed by a conservative force in moving a particle from $\vec{r}_1$ to $\vec{r}_2$ is equal to the negative of the change in potential energy from V_1 to V_2.
- Work performed by the nonconservative forces in carrying a particle from position r_1 to position r_2 is equal to the change in total energy

5.3 d'Alembert Principle The vectorial sum of the external forces and the inertia forces acting on a moving system is zero. Referring to Fig. 6. according to d'Alembert Principle $F + (-m\,a) = 0$ where $-m\,a$ is the inertia force.

5.4 Generalized Principle of d'Alembert:
The Virtual Work performed by the effective forces through infinitesimal virtual displacements compatible with the system constraints is zero

$$\sum_{i=1}^{N}\left(\vec{F}_i - m_i\ \ddot{\vec{r}}\right)\bullet\delta\vec{r}_i = 0$$

5.5 Extended Hamilton's Principle

We can conceive of a 3*N* dimensional space with the axes x_i, y_i, z_i and represent the position of the system of particles in that space and at any time *t* the position of a representative point P with coordinate $x_i(t)$, $y_i(t)$, $z_i(t)$ where i = 1,2,…*N,* the 3N dimensional space is known as the *Configuration Space***.** As time unfolds, the representative point P traces a curve in the configuration space called the *true path, or the Newtonian path, or the dynamical path.* At the same time let us think of a different representative point P' resulting from imagining the system in a slightly different position defined by the virtual displacement δr_i (i = 1,2…*N*). As time changes the point P' traces a curve in the configuration space known as the *Varied Path.*

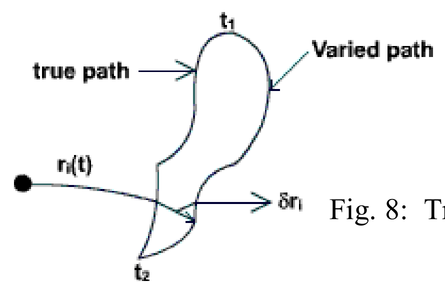


Fig. 8: True and Varied path

Of all the possible varied path, now consider only those that coincide with the true path at the two instants t_1 and t_2 as shown in Fig.8. the *Extended Hamilton's Equation* in terms of Physical coordinates can be given by

$$\int_{t_1}^{t_2}(\delta T + \delta\varpi)dt = 0,\ \delta r_1(t_1) = \delta r_2(t_2) = 0, i = 1,2,....N$$

where δT is the variation in kinetic energy and $\delta\varpi$ is the variation in the work done. But in many cases it is desirable to work with generalized coordinates. As δT and $\delta\varpi$ are independent of coordinates so one can write

$\int_{t_1}^{t_2}(\delta T + \delta\varpi)dt = 0,\ \ \delta q_k(t) = \delta q_k(t) = 0$ where *k* = 1, 2,…*n,* n = no of dof of the system. The extended Principle is very general and can be used for a large variety of systems. The only limitation is that the Virtual displacement must be reversible which implies that the constraint forces must perform no work. Principle cannot be used for system with friction forces.

In general $\delta\varpi = \delta\varpi_c + \delta\varpi_{nc}$ (subscript *c* refers to conservative and *nc* refers to nonconservative). Also $\delta\varpi_c = \delta\omega_c = -\delta V$. Now introducing Lagrangian L = T – V, the extended Hamilton's principle can be written as

$$\int_{t_1}^{t_2} \left(\delta L + \delta \overline{\omega}_{nc}\right) dt = 0, \quad \delta q_k\left(t_1\right) = \delta q_k\left(t_2\right) = 0, \text{ where } k = 1,2,\ldots..n$$

for $\delta \overline{\omega}_{nc} = 0$

$$\boxed{\int_{t_1}^{t_2} \delta L dt = 0, \ \ \delta q_k\left(t_1\right) = \delta q_k\left(t_2\right) = 0} \text{ Hamilton's Principle.}$$

5.6 Lagrange Principle

The Lagrange principle for a damped system can be written as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \left(\frac{\partial L}{\partial q}\right) + \left(\frac{\partial D}{\partial \dot{q}}\right) = Q_k$$

$$Q_k = \sum_i F_l . \frac{\partial \vec{r}_i}{\partial q_k} + \sum_i M_l . \frac{\partial \omega_i}{\partial q_k}, k = 1,2,\ldots.n$$

where L is the Lagrangian given by $L=T-U$, T is the kinetic energy and U is the potential energy of the system. D is the dissipation energy and Q_k is the generalized force. F_i and M_i are the vector representations of the externally applied forces and moments respectively, the index k indicates which external force or moment is being considered, $\boldsymbol{r_i}$ is the position vector to the location where the force is applied, and ω_i is the system angular velocity about the axis along which the considered moment is applied.

Example 6: The system shown in fig. 9 consists of two uniform rigid links of mass m and length L, a massless roller free to move in horizontally and two linear springs of stiffness k_1 and k_2, damper with damping coefficient c and a mass M. Derive the equation of motion either by using extended Hamilton's principle or by Lagrange principle.

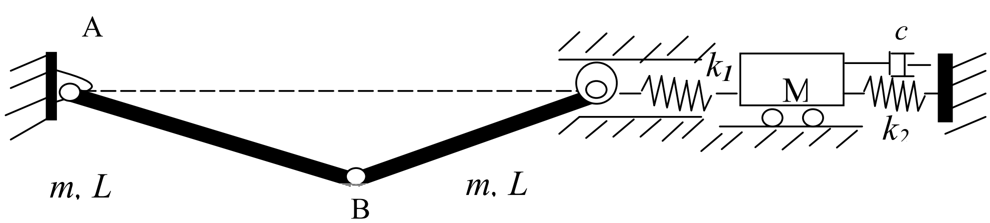


Fig. 9: A system with two rigid links and spring-mass-damper system

Solution:

The system is a two degree of freedom system with generalized coordinates θ and x. The first part ABC can be consider as a slider crank mechanism where motion of any point on the mechanism can be defined in term of θ. Let from initial position OX a small θ rotation is given to link OB.AB. To find the eom first we should find the kinetic and potential energy for which we have to study the kinematics of the system.

<u>Kinematics</u>

Position vector of the roller $\Upsilon_c = \left(L\cos\theta + L\cos\beta\right)\hat{e}$

Position vector of LCg of link1 $\Upsilon_F = \frac{1}{2}\cos\theta\hat{i} - \frac{1}{2}\sin\theta\hat{j}$

Position vector of LCg of link2 $\Upsilon_G = \left(L\cos\theta + \frac{1}{2}\cos\beta\right)\hat{i} - \frac{1}{2}\sin\beta\hat{j}$

Also $L\sin\theta = L\sin\beta \Rightarrow \theta = \beta$

<u>So the velocities</u>

$$\dot{\Upsilon}_F = \frac{1}{2}\sin\theta\dot{\theta}\hat{i} - \frac{1}{2}\cos\theta\dot{\theta}\hat{j} = -\frac{1}{2}\left(\sin\theta\hat{i} + \cos\theta\hat{j}\right)\dot{\theta}$$

$$\dot{\Upsilon}_G = \frac{-3}{2}L\sin\theta\dot{\theta}\hat{i} - \frac{1}{2}\cos\theta\dot{\theta}\hat{j} = -\frac{L\dot{\theta}}{2}\left(3L\sin\theta\hat{i} + \cos\theta\hat{j}\right)$$

While Link AB will rotate about the Pivot Point O, link BC will rotation and translation.

<u>Kinetic energy of the system</u>

<u>= K.E of Link AB + K.E of Link BC + K.E. of mass M.</u>

$$= \frac{1}{2}J_A\dot{\theta}^2 + \frac{1}{2}J_G\dot{\beta}^2 + \frac{1}{2}m_{BC}\dot{\Upsilon}_G.\dot{\Upsilon}_G + \frac{1}{2}\mathrm{M}\dot{x}^2$$

$$= \frac{1}{2}\left(\frac{1}{3}mL^2\right)\dot{\theta}^2 + \frac{1}{2}\left(\frac{1}{12}mL^2\right)\dot{\theta}^2$$

$$+\frac{1}{2}m\left\{-\frac{L}{2}\dot{\theta}\left(3\sin\theta\hat{i} + \cos\theta\hat{j}\right)\right\}\bullet\left\{-\frac{L}{2}\dot{\theta}\left(3\sin\theta\hat{i} + \cos\theta\hat{j}\right)\right\}$$

$$+\frac{1}{2}\mathrm{M}\dot{x}^2$$

$$= \frac{1}{2}\left[\frac{1}{3}mL^2\dot{\theta}^2 + \frac{mL^2}{12}\dot{\theta}^2 + m\left(9\sin^2\theta + \cos^2\theta\right)\dot{\theta}^2\frac{L^2}{4} + \mathrm{M}\dot{x}^2\right]$$

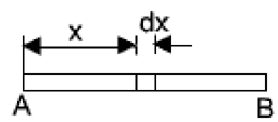
$$= \frac{1}{2}\left[\frac{5mL^2}{12}\dot{\theta}^2 + \frac{mL^2}{4}\left(1 + 8\sin^2\theta\right)\dot{\theta}^2\right] + \frac{1}{2}\mathrm{M}\dot{x}^2$$

$$= \frac{1}{2}\left[\frac{8mL^2}{4}\dot{\theta}^2 + \frac{8mL^2}{4}\sin^2\theta\dot{\theta}^2\right] + \frac{1}{2}\mathrm{M}\dot{x}^2$$

$$= mL^2\dot{\theta}^2 + mL^2\sin^2\theta\dot{\theta}^2 + \frac{1}{2}\mathrm{M}\dot{x}^2$$

$$\Rightarrow \quad \mathrm{T} = mL^2\left(1 + \sin^2\theta\right)\dot{\theta}^2 + \frac{1}{2}\mathrm{M}\dot{x}^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(A)}$$

<u>Now potential energy of the system</u>

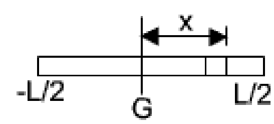


M.I about A

$$J_A = \int\left(\frac{m}{L}dx\right)x^2$$

$$= \frac{m}{L}\int_0^L x^2dx = \left.\frac{mx^3}{3L}\right|_0^L$$

$$= \frac{1}{3}\frac{mL^3}{L} = \frac{1}{3}mL^2$$



$$J_G = \int_{-L/2}^{L/2}\left(\frac{m}{L}dx\right)x^2$$

$$= \left.\frac{m}{L}\frac{x^3}{3}\right|_{-L/2}^{L/2}$$

$$= \frac{m}{3L}\left(\frac{L^3}{8} + \frac{L^3}{8}\right)$$

$$= \frac{2mL^3}{24L} = \frac{mL^2}{12}$$

Spring K_1 undergoes a displacement of

$$\left[x-\left(2L-\Upsilon_c.\hat{i}\right)\right]\hat{i}$$

$$=\left[x-\left(2L-2L\cos\theta\right)\right]\hat{i}=\left[x-2L\left(1-\cos\theta\right)\right]\hat{i}$$

Spring K_2 undergoes a displacement of $x\hat{i}$

Total P.E $=-mg.\frac{L}{2}\sin\theta-mg\frac{L}{2}\sin\theta+\frac{1}{2}K_1\left\{x-2L\left(1-\cos\theta\right)\right\}^2+\frac{1}{2}K_2x^2$

$$=-mgL\sin\theta+\frac{1}{2}K_1\left\{x^2+4L^2\left(1-\cos\theta\right)^2-4xL\left(1-\cos\theta\right)\right\}+\frac{1}{2}K_2x^2$$

damping energy $=\frac{1}{2}c\dot{x}^2$

<u>Hamilton's Principle</u>

$$L=T-U=mL^2\left(1+\sin^2\theta\right)\dot{\theta}^2+\frac{1}{2}M\dot{x}^2$$

$$-\left[mgL\sin\theta+\frac{1}{2}K_1x^2-2K_1Lx\left(1-\cos\theta\right)+2K_1L^2\left(1-\cos\theta\right)^2+\frac{1}{2}K_2x^2\right]$$

$$\int_{t_1}^{t_2}\left(\delta L+\overline{\delta\omega_{nc}}\right)dt=0 \qquad \delta x=0,\ \delta\theta=0\ \ at\ \ t=t_1,t_2$$

$$\delta L=mL^2\left(1+\sin^2\theta\right)2\dot{\theta}\delta\dot{\theta}+mL^2\left(2\sin\theta\cos\theta.\delta\theta\right)\dot{\theta}^2+mgL\cos\theta.\delta\theta$$

$$-2K_1L^2\left(2\left(1-\cos\theta\right)\sin\theta.\delta\theta\right)+2K_1L\delta x\left(1-\cos\theta\right)+2K_1Lx\sin\theta.\delta\theta$$

$$-\frac{1}{2}K_1 2x\delta x+\frac{1}{2}M2\dot{x}\delta\dot{x}-\frac{1}{2}K_2 2x\delta x$$

$$\int\delta\omega_{nc}=-\int c\dot{x}\delta x dt$$

so, $\int_{t_1}^{t_2}\left(\delta L+\overline{\delta\omega_{nc}}\right)dt=\int_{t_1}^{t_2}2mL^2\left(1+\sin^2\theta\right)\dot{\theta}\delta\dot{\theta}dt$

$$=2mL^2\int_{t_1}^{t_2}\left(1+\sin^2\theta\right)\dot{\theta}\frac{d}{dt}\left(\delta\theta\right)dt$$

$$=2mL^2\left[\left(1+\sin^2\theta\right)\dot{\theta}\delta\theta\Big|_{t_1}^{t_2}-\int_{t_1}^{t_2}\left\{\left(1+\sin^2\theta\right)\ddot{\theta}+\left(2\sin\theta\cos\theta\right)\dot{\theta}\right\}\delta\theta dt\right]$$

and $\int_{t_1}^{t_2}M\dot{x}\delta\dot{x}dt=\int_{t_1}^{t_2}M\dot{x}\frac{d}{dt}\left(\delta x\right)dt=M\dot{x}\delta x\Big|_{t_1}^{t_2}-\int_{t_1}^{t_2}M\ddot{x}\delta x dt$

$$\int_{t_1}^{t_2}c\dot{x}\delta\dot{x}=\int_{t_1}^{t_2}c\dot{x}\frac{d}{dt}\left(\delta x\right)dt=c\dot{x}\delta x\Big|_{t_1}^{t_2}-\int_{t_1}^{t_2}c\ddot{x}\delta x dt$$

so, $\int_{t_1}^{t_2}\left(\delta L+\overline{\delta\omega_{nc}}\right)dt$

$$= \int_{t_1}^{t_2} \begin{bmatrix} -2mL^2 \left\{ \left(1+\sin^2\theta\right)\ddot{\theta} + 2\sin\theta\cos\theta\ \dot{\theta} \right\} + mL^2\left(2\sin\theta\cos\theta\right)\dot{\theta}^2 \\ +mgL\cos\theta - 4\mathrm{K}_1 L^2 \sin\theta\left(1-\cos\theta\right) + 2\mathrm{K}_1 L x\sin\theta \end{bmatrix} \delta\theta.dt$$

$$+\int_{t_1}^{t_2}\left[2\mathrm{K}_1 L\left(1-\cos\theta\right) - \mathrm{K}_1 x - \mathrm{K}_2 x - M\ddot{x} - c\dot{x}\right]\delta x.dt = 0$$

so equation of motions are

$$2mL^2\left(1+\sin^2\theta\right)\ddot{\theta} + 4mL^2\sin\theta\cos\theta\ \dot{\theta} - 2mL^2\sin\theta\cos\theta\ \dot{\theta}^2$$
$$-mgL\cos\theta + 4\mathrm{K}_1 L^2\sin\theta\left(1-\cos\theta\right) - 2\mathrm{K}_1 L x\sin\theta = 0 \text{ ....(a)}$$

$$M\ddot{x} + c\dot{x} + \mathrm{K}_1 x + \mathrm{K}_2 x - 2\mathrm{K}_1 L\left(1-\cos\theta\right) = 0 \text{ ....(b)}$$

Now taking θ to be small, $\sin\theta \simeq \theta$, $\cos\theta = 1$, $\theta^2 = 0$

$$2mL^2\left(1+\theta^2\right)\ddot{\theta} + 4mL^2\theta\ \dot{\theta} - 2mL^2\theta\ \dot{\theta}^2 - mgL - 2\mathrm{K}_1 L x\theta = 0$$
$$M\ddot{x} + c\dot{x} + \mathrm{K}_1 x + \mathrm{K}_2 x = 0$$

$$\boxed{\begin{array}{l} 2mL^2\ddot{\theta} - 2\mathrm{K}_1 L\theta x - mgL + 4mL^2\theta\ \dot{\theta} \\ M\ddot{x} + c\dot{x} + \mathrm{K}_1 x = 0 \end{array}}$$

$$\begin{pmatrix} M & 0 \\ 0 & 2mL^2 \end{pmatrix}\begin{pmatrix} \ddot{x} \\ \ddot{\theta} \end{pmatrix} + \begin{pmatrix} c & 0 \\ 0 & 4mL^2\theta \end{pmatrix}\begin{pmatrix} \dot{x} \\ \dot{\theta} \end{pmatrix} + \begin{pmatrix} K & 0 \\ & \end{pmatrix}\begin{pmatrix} x \\ \theta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

<u>Lagrange equation for a Dissipative System</u>

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} + \frac{\partial F_d}{\partial \dot{q}_k} = Q_{nc}$$

Dissipative Energy = $\frac{1}{2}c\dot{x}^2$

Other non conservative force $Q_{nc}=0$

$q_1 = \theta$, $q_2 = x$

$$L = mL^2\left(1+\sin^2\theta\right)\dot{\theta}^2 + \frac{1}{2}M\dot{x}^2$$
$$-\left[-mgL\sin\theta + \frac{1}{2}\mathrm{K}_1 x^2 - 2\mathrm{K}_1 L x\left(1-\cos\theta\right) + 2\mathrm{K}_1 L^2\left(1-\cos\theta\right)^2 + \frac{1}{2}\mathrm{K}_2 x^2\right]$$

$$\frac{\partial L}{\partial \dot{\theta}} = mL^2\left(1+\sin^2\theta\right)2\dot{\theta}$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = 2mL^2\left(1+\sin^2\theta\right)\ddot{\theta} + 2mL^2\dot{\theta}.2\sin\theta\cos\theta\ \dot{\theta}$$

$$\Rightarrow \frac{\partial L}{\partial \theta} = mL^2\dot{\theta}^2\left(\sin\theta\cos\theta\right) + mgL\cos\theta + 2\mathrm{K}_1Lx\sin\theta - 2\mathrm{K}_1L^2 2\left(1-\cos\theta\right)\sin\theta$$

$$\frac{\partial L}{\partial \dot{x}} = \frac{1}{2}M\dot{x}.2, \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = M\ddot{x}$$

$$\frac{\partial L}{\partial x} = -\frac{1}{2}2\mathrm{K}_1x + 2\mathrm{K}_1L\left(1-\cos\theta\right) - \frac{1}{2}\mathrm{K}_2 2x$$

$$\frac{\partial F_D}{\partial \dot{\theta}} = 0, \quad \frac{\partial F_D}{\partial \dot{x}} = \frac{1}{2}c2\dot{x} = c\dot{x}$$

so eom

$$2mL^2\left(1+\sin^2\theta\right)\ddot{\theta} + 4mL^2\dot{\theta}^2\sin\theta\cos\theta - 2mL^2\dot{\theta}^2\sin\theta\cos\theta - mgL\cos\theta$$

$$-2\mathrm{K}_1Lx\sin\theta + 4\mathrm{K}_1L^2\left(1-\cos\theta\right)\sin\theta = 0$$

$$\boxed{M\ddot{x} + \mathrm{K}_1x - 2\mathrm{K}_1L\left(1-\cos\theta\right) + \mathrm{K}_2x + c\dot{x} = 0} \quad \text{………………………(a)}$$

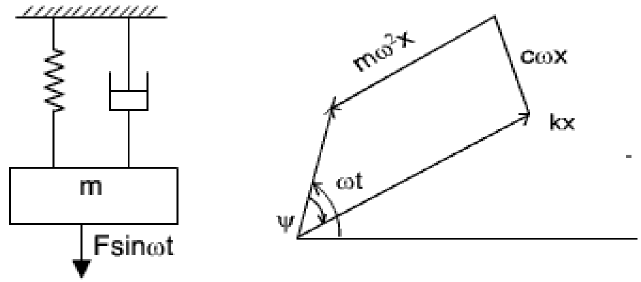
$$\boxed{\begin{aligned}&2mL^2\left(1+\sin^2\theta\right)\ddot{\theta} + 2mL^2\dot{\theta}^2\sin\theta\cos\theta - mgL\cos\theta - 2\mathrm{K}_1Lx\sin\theta + \\ &4\mathrm{K}_1L^2\left(1-\cos\theta\right)\sin\theta = 0\end{aligned}}$$

……….(b)

5.7 RESPONSE FOR LINEAR SYSTEMS

After deriving the equation of motion, if the equation is a nonlinear one, one may either linearize the equation to solve the linearized equation, or one may directly go for a nonlinear analysis. Also, for a given system parameters, the response can be obtained by numerical integration technique such as Runge-Kutte method. For solving nonlinear vibration problems one may use (i) straight forward expansion, (ii) the Lindstedt-Poincare method, (iii) the method of multiple scales, (iv) the method of harmonic balance, (v) the method of averaging, etc. Here two problems one for linear system and other for a nonlinear system is explained with the help of examples. The nonlinear system is solved using the method of multiple scales.

Example 7:
Force Vibration of Single Degree of Freedom Systems with Harmonic Oscillation:



$$\omega_n = \sqrt{\frac{k}{m}}$$

$c_c = 2m\omega_n$ = Critical damping

$\zeta = \frac{c}{c_c}$ = damping factor

$$\frac{c\omega}{k} = \frac{c}{c_c} * \frac{c_c}{k} = 2\zeta \frac{\omega}{\omega_n}$$

$$\frac{c}{m} = 2\zeta\omega_n$$

Figure 10 (a) Spring-mass damper system (b) Force polygon

$$\mathrm{X} = \frac{F_0}{\sqrt{\left(k - m\omega^2\right)^2 + \left(c\omega\right)^2}}$$

$$\phi = \tan^{-1}\frac{c\omega}{k - m\omega^2}$$

$$\mathrm{X}(t) = x_1 e^{-\zeta\omega_n t}\sin\left(\sqrt{1-\zeta^2}\,\omega_n t + \phi_1\right) + \frac{F_0}{k}\frac{\sin(\omega t - \phi)}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}}$$

6. Qualitative Analysis of Nonlinear Systems

For the nonlinear system $\ddot{u} + f(u) = 0$

Upon integrating one may write

$$\int (\dot{u}\ddot{u} + \dot{u}f(u))dt = h$$

or, $\frac{1}{2}\dot{u}^2 + F(u) = h, \quad F(u) = \int f(u)du$

This represents that the sum of the kinetic energy and potential energy of the system is constant. Hence, for particular energy level h, the system will be under oscillation, if the potential energy $F(u)$ is less than the total energy h . From the above equation, one may plot the phase portrait or the trajectories for different energy level and study qualitatively about the response of the system.

Example 8 Perform qualitative analysis to study the response of the dynamic system

$$\ddot{x} + x - 0.1x^3 = 0$$

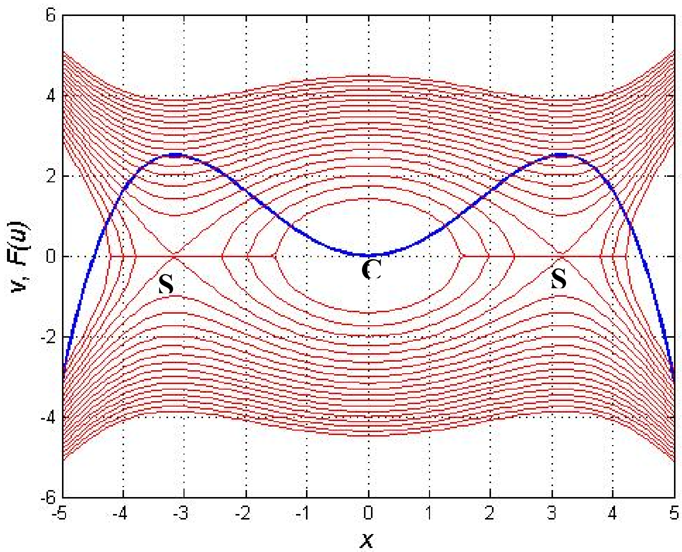


Fig. 11 Potential well and phase portrait showing saddle point and center corresponding to maximum and minimum potential energy.

Solution: For this system $F(x)=\int f(x)dx=\int(x-0.1x^3)dx=\frac{1}{2}x^2-\frac{1}{40}x^4$

The above figure shows the variation of potential energy $F(x)$ with x. It has its optimum values corresponding to $x=0$ or $\pm\sqrt{20}$ While x equal to zero represents the system with minimum potential energy, the other two points represent the points with maximum potential energy. Now by taking different energy level h, one may find the relation between the velocity v and displacement x as

$$v=\dot{x}=\sqrt{2(h-F(x))}=2\sqrt{2(h-(0.5x^2-0.025x^4))}$$

Now by plotting the phase portrait one may find the trajectory which clearly depicts that the motion corresponding to maximum potential energy is unstable and the bifurcation point is of saddle-node type (marked by point S) and the motion corresponding to the minimum potential energy is stable center type (marked by point C).

Lagrange and Dirichlet Theorem If the potential energy has an isolated minimum at an equilibrium point, the equilibrium state is stable.
Liapunov Theorem: If the potential energy at an equilibrium point is not a minimum, the equilibrium state is unstable

At saddle point $\frac{d^2F}{du^2}=\frac{df}{du}<0$, At center $\frac{d^2F}{du^2}=\frac{df}{du}>0$

The motion is oscillatory in the neighborhood of center

7. Stability Analysis

For a dynamic system, one may write the governing differential equation of motion as a set of first order differential equation or one may reduce the governing equation of motion by applying perturbation method to the following form.

$$\dot{x}=F(x,M) \quad (1)$$

In this equation M represents the control parameters or the system parameters. The steady state response of this system can be obtained by substituting $\dot{x}=0$, and solving the resulting nonlinear algebraic/transcendental equation. To obtain the stability of the steady state fixed point response, one may perturb or give a small disturbance to the above mentioned equilibrium point and study its behaviour. While for a stable equilibrium point, the system return backs to the original position, for unstable system, after perturbation, the system response grows. Hence, to study stability of the system one uses the following steps.

Considering x_0 as the equilibrium point, substitute $x(t) = x_0 + y(t)$ in equation (1). The resulting equation will be

$$\dot{y} = F\left(x_0 + y, M_0\right) + D_x F\left(x_0; M_0\right) y + O\left(\| y \|^2\right) \Rightarrow \dot{y} = D_x F\left(x_0; M_0\right) y \equiv Ay \quad (2)$$

$$\text{where, } A = \begin{bmatrix} \frac{dF_1}{dx_1} & \frac{dF_1}{dx_2} & \cdots & \frac{dF_1}{dx_n} \\ \frac{dF_2}{dx_1} & \frac{dF_2}{dx_2} & \cdots & \frac{dF_2}{dx_n} \\ . & . & . & . \\ \frac{dF_n}{dx_1} & \frac{dF_n}{dx_2} & \cdots & \frac{dF_n}{dx_n} \end{bmatrix} \quad (3)$$

This matrix is known as the Jacobian matrix and the eigenvalues of the constant matrix A provides the information about the local stability of the fixed point $x0$.

8. Classification of fixed point response

- Hyperbolic fixed point: when all of the eigenvalues of A have nonzero real parts it is known as hyperbolic fixed point.
- Sink: If all of the eigenvalues of A have negative real part. The sink may be of stable focus if it has nonzero imaginary parts and it is of stable node if it contains only real eigenvalues which are negative.
- Source: If one or more eigenvalues of A have positive real part. Here, the system is unstable and it may be of unstable focus or unstable node.
- Saddle point: when some of the eigenvalues have positive real parts while the rest of the eigenvalues have negative.
- Marginally stable: If some of the eigenvalues have negative real parts while the rest of the eigenvalues have zero real parts

In nonlinear systems, while plotting the frequency response curves of the system by changing the control parameters, one may encounter the change of stability or change in the number of equilibrium points. These points corresponding to which the number or nature of the equilibrium point changes, are known as bifurcation points. For fixed point response, they may be divided into static or dynamic bifurcation points depending on the nature of the eigenvalues of the system. If the eigen values are plotted in a complex plane with their real and imaginary parts along *X* and *Y* directions, a static bifurcation occurs, if with change in the control parameter, an eigenvalue of the Jacobian matrix crosses the origin of the complex plane. In case of dynamic bifurcation, a pair of complex conjugate eigenvalues crosses the imaginary axis with change in control parameter of the system. Hence, in this case the resulting solution is stable or unstable periodic type.

Saddle-node Bifurcation

The normal form for a generic saddle-node bifurcation of a fixed point is $\dot{x} = F(x;\mu) = \mu - x^2$ where, μ is the control parameter. In this case the equilibrium points are $x = \pm\sqrt{\mu}$ and the eigenvalue is -2x which change its sign at $x_0 = 0$. For positive value of x_0 the response is stable and for negative value of x_0 the response is unstable.

Example 9 For a typical dynamic system the frequency-amplitude relation is given by the following equation.

$$\dot{a} = -\zeta a - \bar{\omega}_1^2\left(\frac{1}{8}\alpha_4 a^2 + \frac{1}{2}\alpha_5\right)\sin\gamma$$

$$a\dot{\gamma} = a\sigma - \frac{3}{8}K a^3 - \bar{\omega}_1^2\left(\frac{3}{8}\alpha_4 a^2 + \frac{1}{2}\alpha_5\right)\cos\gamma.$$

Here, $\bar{\omega}_1 = 1 + \varepsilon\sigma$. $\zeta, \alpha_4, \alpha_5, k$ are fixed system parameters. The saddle node bifurcation points have been shown in Fig. 12. (b)

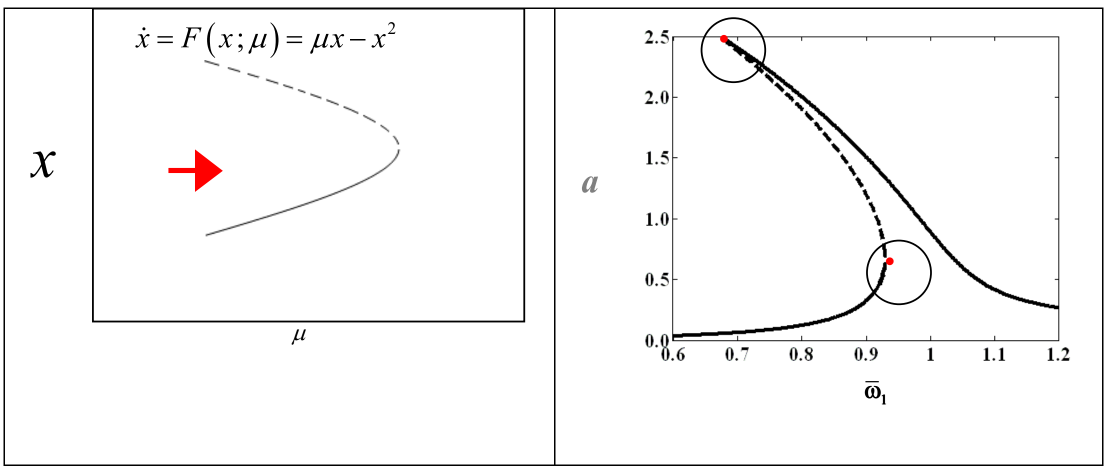


Fig.12: Saddle-node bifurcation point corresponding to (a) $\dot{x} = F(x;\mu) = \mu - x^2$, (b) example

Pitchfork bifurcation: The normal form for a generic pitchfork bifurcation of a fixed point is

Trans-critical bifurcation: The normal form for a generic pitchfork bifurcation of a fixed point is $\dot{x} = F(x;\mu) = \mu x - x^2$

Hopf bifurcation: The normal form for a generic Hopf bifurcation of a fixed point is

$$\dot{x} = \mu x - \omega y + (\alpha x - \beta y)(x^2 + y^2)$$

$$\dot{y} = \mu x - \omega y + (\alpha x - \beta y)(x^2 + y^2)$$

It may be noted that while supercritical pitchfork and Hopf bifurcation, respectively results in stable fixed-point and periodic responses, the sub-critical pitchfork and Hopf bifurcation, respectively results in unstable fixed-point and periodic responses. Hence, these sub critical bifurcation points are dangerous bifurcation points.

In case of nonlinear vibration many phenomena such as jump-up, jump-down, saturation, multi-stable region along with different types of responses such as fixed-point, periodic, quasi-periodic and chaotic are observed. Many bifurcation phenomena such as sub and super critical pitchfork, Hopf, saddle-point, period doubling etc are observed. One may observe different type of crisis phenomena in chaotically modulated system.

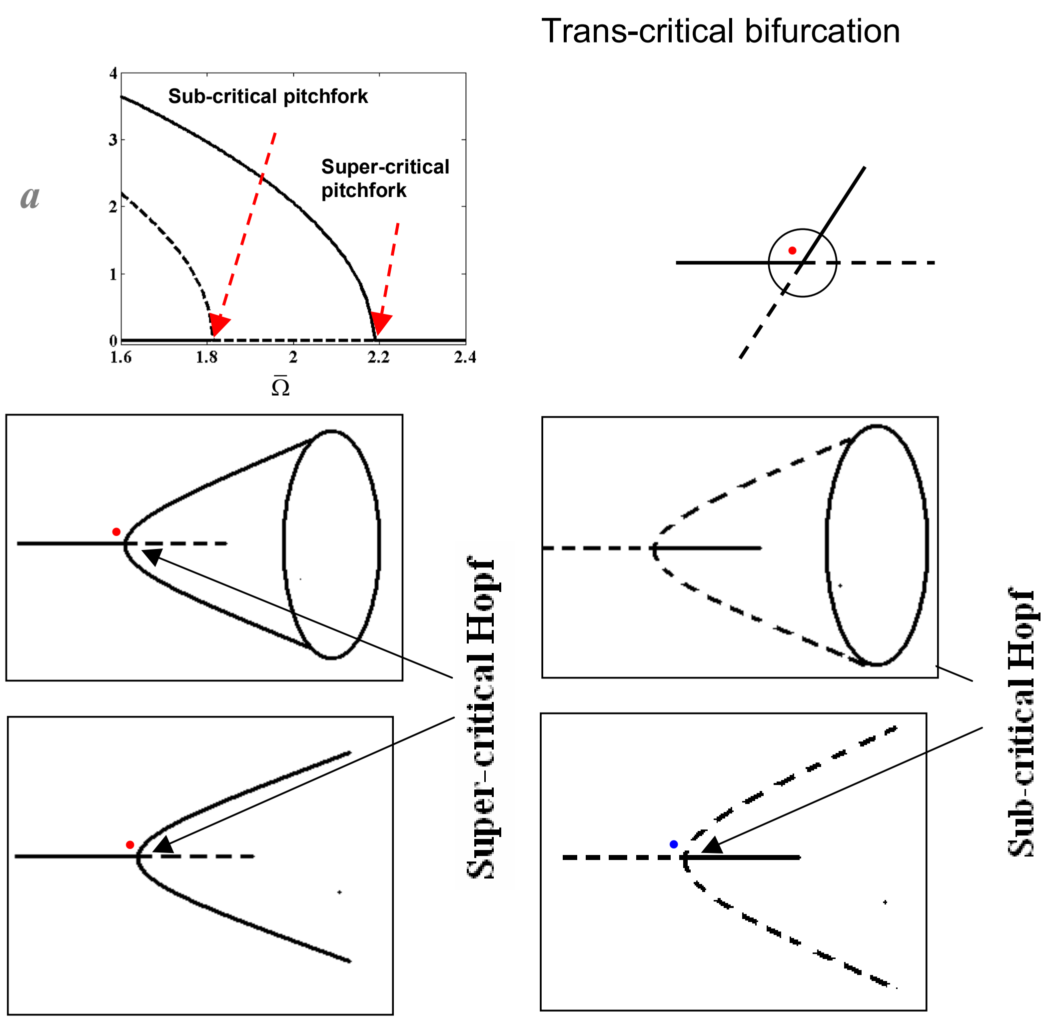


Fig. 13 (a) Pitchfork bifurcation, (b) trans-critical bifurcation, (c) super-critical Hopf bifurcation, (d) sub-critical Hopf bifurcation.

Summary

In this lecture note the derivation of the equation of motion of dynamical systems has been illustrated by starting with a vector approach, and taking both inertia and energy based principles. For nonlinear systems, a qualitative approach has been explained and the classification of stability and bifurcation of the fixed-point responses have been illustrated with the help of examples. A number of references have been given in the reference section for the interested reader, which may be referred to know the different perturbation methods used to study nonlinear systems.

References:

1. L. Meirovitch, Elements of Vibration Analysis, McGraw Hill, Second edition,1986.
2. L. Meirovitch, Principles and Techniques of Vibrations, Prentice Hall International (PHIPE), New Jersey, 1997.
3. A. H. Nayfeh and D. T. Mook, 1979 *Nonlinear oscillations*, New York, Willey Interscience.
4. Z. Rahman and T. D. Burton, 1989 *Journal of Sound and Vibration* **133,** 369-379. On higher order method of multiple scales in nonlinear oscillations-periodic steady state response.
5. H. Nayfeh and D. T. Mook, Nonlinear Oscillations, Wiley, 1979
6. A.H. Nayfeh and B. Balachandran, Applied Nonlinear Dynamics, Wiley,1995
7. K. Huseyin and R. Lin: An Intrinsic multiple- time-scale harmonic balance method for nonlinear vibration and bifurcation problems, International Journal of Nonlinear Mechanics, 26(5),727-740,1991
8. J. J. Wu. A generalized harmonic balance method for forced nonlinear oscillations: the subharmonic cases. Journal of Sound and Vibration, 159(3), 503-525,1992
9. J. J. Wu and L. C. Chien, Solution to a general forced nonlinear oscillations problem. Journal of Sound and Vibration, 185(2),247-264, 1995 Multiple time scales harmonic balance.
10. S. L. Lau, Y. K. Cheung and S. Y. Wu, Incremental harmonic balance method with multiple time scales for aperiodic vibration of nonlinear systems. Journal of applied Mechanics, ASME, 50(4), 871-876, 1983.
11. S. H. Chen and Y. K. Cheung, A modified Lindstedt-Poincare' method for a strongly nonlinear two-degree-of-freedom system. Journal of Sound and Vibration, 193(4), 751-762, 1996.