FDTD: An Introduction

Prof. Rakhesh Singh Kshetrimayum
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- How do one visualize wave?
  - How can one observe wave propagation?
- Radio waves can’t be observed in nature
  - But one can write some EM codes (simulation) and observe wave behavior
- One popular choice for this is Finite Difference Time Domain method
  - Also referred to as FDTD
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- What is FDTD?
- Let us start with 4 Maxwell’s equations

1. \( \nabla \cdot \vec{E} = \frac{\rho_v}{\varepsilon} \)

2. \( \nabla \cdot \vec{B} = 0 \)

3. \( \nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \)

4. \( \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \)

Maxwell curl equations (why?)
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• Basically, we replace spatial & time derivatives in the two Maxwell’s curl equations
  • by central finite difference approximation
• What is central finite difference approximation of derivatives?
• Consider a function $f(x)$, its derivative at $x_0$ from central finite difference approximation is given by

\[
\frac{df(x_0)}{dx} = f'(x_0) \approx \frac{f\left(x_0 + \frac{\Delta x}{2}\right) - f\left(x_0 - \frac{\Delta x}{2}\right)}{\Delta x}
\]
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- The time-dependent and source free ($\vec{J} = 0$) Maxwell’s curl equations in a medium with $\varepsilon = \varepsilon_0 \varepsilon_r$ and $\mu = \mu_0 \mu_r$

3 equations

\[
\frac{\partial \vec{E}}{\partial t} = \frac{1}{\varepsilon_0 \varepsilon_r} \nabla \times \vec{H}; \quad \frac{\partial \vec{H}}{\partial t} = -\frac{1}{\mu_0 \mu_r} \nabla \times \vec{E}
\]

- For Cartesian coordinate systems,
  - expanding the curl equations,
  - equating the vector components,
- we have 6 equations
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- Expanding the first vector curl equation

\[
\frac{\partial \vec{E}}{\partial t} = \frac{1}{\varepsilon_0 \varepsilon_r} \nabla \times \vec{H}
\]

- we get 3 equations

\[
\frac{\partial E_x}{\partial t} = \frac{1}{\varepsilon_0 \varepsilon_r} \left( \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} \right)
\]
\[
\frac{\partial E_y}{\partial t} = \frac{1}{\varepsilon_0 \varepsilon_r} \left( \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} \right)
\]
\[
\frac{\partial E_z}{\partial t} = \frac{1}{\varepsilon_0 \varepsilon_r} \left( \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \right)
\]
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- Expanding the second vector curl equation

\[
\frac{\partial \vec{H}}{\partial t} = -\frac{1}{\mu_0 \mu_r} \nabla \times \vec{E}
\]

- We get 3 equations

\[
\frac{\partial H_x}{\partial t} = \frac{1}{\mu_0 \mu_r} \left( \frac{\partial E_y}{\partial z} - \frac{\partial E_z}{\partial y} \right)
\]
\[
\frac{\partial H_y}{\partial t} = \frac{1}{\mu_0 \mu_r} \left( \frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z} \right)
\]
\[
\frac{\partial H_z}{\partial t} = \frac{1}{\mu_0 \mu_r} \left( \frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x} \right)
\]
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As we know that FDTD is a time-domain solver
The question is how do we solve those 6 equations above?

1. 1-D FDTD update equations

• For 1-D case (a major simplification), we can consider

• (a) Linearly polarized wave along x-axis
  • exciting an electric field which has $E_x$ only ($E_y = E_z = 0$)

• (b) propagation along z-axis
  • no variation in the x-y plane, i.e. $\partial/\partial x = 0$ and $\partial/\partial y = 0$
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\[
\begin{align*}
\frac{\partial E_x}{\partial t} &= \frac{1}{\varepsilon_0 \varepsilon_r} \left( \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} \right) \\
\frac{\partial E_y}{\partial t} &= \frac{1}{\varepsilon_0 \varepsilon_r} \left( \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} \right) \\
\frac{\partial E_z}{\partial t} &= \frac{1}{\varepsilon_0 \varepsilon_r} \left( \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \right) \\
\frac{\partial H_x}{\partial t} &= \frac{1}{\mu_0 \mu_r} \left( \frac{\partial E_y}{\partial z} - \frac{\partial E_z}{\partial y} \right) \\
\frac{\partial H_y}{\partial t} &= \frac{1}{\mu_0 \mu_r} \left( \frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z} \right) \\
\frac{\partial H_z}{\partial t} &= \frac{1}{\mu_0 \mu_r} \left( \frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x} \right)
\end{align*}
\]
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- Then, the above 6 equations from 2 Maxwell’s curl equations
- reduce to 2 equations for 1-D FDTD
  \[
  \frac{\partial E_x}{\partial t} = -\frac{1}{\varepsilon_0 \varepsilon_r} \left( \frac{\partial H_y}{\partial z} \right) \quad \frac{\partial H_y}{\partial t} = -\frac{1}{\mu_0 \mu_r} \left( \frac{\partial E_x}{\partial z} \right)
  \]
- We want to solve these equations at different locations and time in solution space (observe fields at different place and time)
- Hence we need to discretize both in time and space
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space discretization size $\Delta z$ running index $k$

Fig. 1(a) Discretizations in 1-D space

time discretization size $\Delta t$ running index $n$

Fig. 1(b) Discretization in time
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- Notations:
  - \( n \) is the time index and
  - \( k \) is the spatial index,

- How to decide space and time discretization size?
- How do one determine the values of \( \Delta z \) and \( \Delta t \)?
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- Locate positions $z = k\Delta z$
- Usually $\Delta z$ is taken as 10-15 per wavelength
  - gives accurate results
- times $t = n\Delta t$ and
- $\Delta t$ (dependent on $\Delta z$) chosen from Courant stability criterion
  - gives stable FDTD solution (will be discussed later)
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- Let us discretize the first equation:
  \[
  \frac{\partial E_x}{\partial t} = -\frac{1}{\varepsilon_0\varepsilon_r} \left( \frac{\partial H_y}{\partial z} \right)
  \]

- the central difference approximations for both the temporal and spatial derivatives are obtained at
  - \( z = k\Delta z \), (space discretization size and index)
  - \( t = n\Delta t \) (time discretization size and index)

\[
\frac{E_x(k,n+\frac{1}{2}) - E_x(k,n-\frac{1}{2})}{\Delta t} = -\frac{1}{\varepsilon_0\varepsilon_r} \frac{H_y(k+\frac{1}{2},n) - H_y(k-\frac{1}{2},n)}{\Delta z}
\]
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- For the second equation:
  \[ \frac{\partial H_y}{\partial t} = -\frac{1}{\mu_0\mu_r} \left( \frac{\partial E_x}{\partial z} \right) \]

- the central difference approximations for both the temporal and spatial derivatives are obtained at
- at \((z + \Delta z/2, t + \Delta t/2)\)
- (increment all the time and space steps by 1/2):

\[
\frac{H_y(k + \frac{1}{2}, n + 1) - H_y(k + \frac{1}{2}, n)}{\Delta t} = -\frac{1}{\mu_0\mu_r} \frac{E_x(k + 1, n + \frac{1}{2}) - E_x(k, n + \frac{1}{2})}{\Delta z}
\]
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- The above equations can be rearranged as a pair of ‘computer update equations’,
- which can be repeatedly updated in loop,
- to obtain the next instant time values of $E_x(k, n+1/2)$ and $H_y(k + \frac{1}{2}, n+1)$
  - from the previous instant time values as follows

\[
E_x(k, n+\frac{1}{2}) = E_x(k, n-\frac{1}{2}) - \frac{1}{\varepsilon_0\varepsilon_r}\frac{\Delta t}{\Delta z}[H_y(k + \frac{1}{2}, n) - H_y(k - \frac{1}{2}, n)]
\]

\[
H_y(k + \frac{1}{2}, n+1) = H_y(k + \frac{1}{2}, n) - \frac{1}{\mu_0\mu_r}\frac{\Delta t}{\Delta z}[E_x(k + 1, n + \frac{1}{2}) - E_x(k, n + \frac{1}{2})]
\]

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Fig. 2 Interleaving of E and H fields
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- Interleaving of the E and H fields in space and time in the FDTD formulation
- to calculate $E_x(k)$,
  - the neighbouring values of $H_y$ at $k-1/2$ and $k+1/2$ of the previous time instant are needed
- Similarly,
- to calculate $H_y(k+1/2)$, for instance,
  - the neighbouring values of $E_x$ at $k$ and $k+1$ of the previous time instants are needed
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- In the above equations,
  - \( \varepsilon_0 \) and \( \mu_0 \) differ by several orders of magnitude,
  - \( E_x \) and \( H_y \) will differ by several orders of magnitude
  - We also know that ratio of electric field and magnetic field for plane waves is \( \frac{120}{\pi} \)
- Numerical error is minimized by making the following change of variables as

\[
E'_x = \sqrt{\frac{\varepsilon}{\mu}} E_x
\]
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• Hence,

\[
\frac{1}{\sqrt{\varepsilon / \mu}} E_x'(k, n + \frac{1}{2}) = \frac{1}{\sqrt{\varepsilon / \mu}} E_x'(k, n - \frac{1}{2}) - \frac{1}{\varepsilon / \Delta z} \Delta t\left[H_y(k + \frac{1}{2}, n) - H_y(k - \frac{1}{2}, n)\right]
\]

\[
\Rightarrow E_x'(k, n + \frac{1}{2}) = E_x'(k, n - \frac{1}{2}) - \frac{1}{\sqrt{\varepsilon / \mu}} \frac{\Delta t}{\varepsilon / \Delta z}\left[H_y(k + \frac{1}{2}, n) - H_y(k - \frac{1}{2}, n)\right]
\]

\[
\Rightarrow E_x'(k, n + \frac{1}{2}) = E_x'(k, n - \frac{1}{2}) - \frac{1}{\sqrt{\mu / \varepsilon}} \frac{\Delta t}{\Delta z}\left[H_y(k + \frac{1}{2}, n) - H_y(k - \frac{1}{2}, n)\right]
\]

\[
\Rightarrow E_x'(k, n + \frac{1}{2}) = E_x'(k, n - \frac{1}{2}) + \frac{1}{\sqrt{\mu / \varepsilon}} \frac{\Delta t}{\Delta z}\left[H_y(k - \frac{1}{2}, n) - H_y(k + \frac{1}{2}, n)\right]
\]
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- Similarly,

\[
H_y\left(k+\frac{1}{2},n+1\right) = H_y\left(k+\frac{1}{2},n\right) - \frac{1}{\sqrt{\varepsilon \mu}} \Delta t \left[ E'_x\left(k+1,n+\frac{1}{2}\right) - E'_x\left(k,n+\frac{1}{2}\right) \right]
\]

\[
\Rightarrow H_y\left(k+\frac{1}{2},n+1\right) = H_y\left(k+\frac{1}{2},n\right) - \frac{1}{\sqrt{\varepsilon \mu}} \Delta t \left[ E'_x\left(k+1,n+\frac{1}{2}\right) - E'_x\left(k,n+\frac{1}{2}\right) \right]
\]

\[
\Rightarrow H_y\left(k+\frac{1}{2},n+1\right) = H_y\left(k+\frac{1}{2},n\right) + \frac{1}{\sqrt{\mu \varepsilon}} \Delta t \left[ E'_x\left(k,n+\frac{1}{2}\right) - E'_x\left(k+1,n+\frac{1}{2}\right) \right]
\]
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Courant stability criteria:

- Courant stability criteria dictates the
  - relationship between the time increment $\Delta t$ with respect to space increment $\Delta z$
  - in order to have a stable FDTD solution of the electromagnetic problems
- In isotropic media, an electromagnetic wave propagates a distance of one cell in time $\Delta t = \Delta z / v_p$,
  - where $v_p$ is the phase velocity
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- This limits the maximum time step
- This equation implies that an EM wave cannot be allowed
  - to move more than a space cell during a time step
- Otherwise, FDTD will start diverging
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- If we choose $\Delta t > \Delta z/v_p$,
  - the distance moved by the EM wave over the time interval $\Delta t$ will be more than $\Delta z$,
  - the EM wave will leave out the next node/cell and
  - FDTD cells are not causally interconnected and
- hence, it leads to instability
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Fig. 3 (a) 2-D discretizations
Fig. 3 (b) Plane wave front propagation along $\theta=0^0$ and $\theta=45^0$
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- Propagation at $\theta=45^0$
- Wavefront jumps from one row of nodes to the next row,
- the spacing between the consecutive rows of nodes is $\frac{\Delta}{\sqrt{2}}$

- Similarly for 3-D case $\frac{\Delta}{\sqrt{3}}$
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- In the case of a 2-D simulation,
- we have to allow for the propagation in the diagonal direction,
- which brings the time requirement to \( \Delta t = \frac{\Delta z}{\sqrt{2} v_p} \)
- Obviously, three-dimensional simulation requires \( \Delta t = \frac{\Delta z}{\sqrt{3} v_p} \)
- We will use in all our simulations a time step \( \Delta t \) of

\[
\Delta t = \frac{\Delta z}{2 \cdot v_p}
\]
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- where \( v_p \) is the phase velocity,
- which satisfies the requirements in 1-D, 2-D and 3-D for all media (\( \sqrt{2} \approx 1.414 < \sqrt{3} \approx 1.7321 < 2 \))
- In 3D case, it may be more appropriate to modify the above stability criteria as

\[
\Delta t = \frac{\min (\Delta x, \Delta y, \Delta z)}{2 \cdot v_p}
\]

- Using the above relation, we may simplify for 1-D FDTD as

\[
\frac{1}{\sqrt{\varepsilon \mu}} \frac{\Delta t}{\Delta z} = v_p \cdot \frac{\Delta t}{\Delta z} = \frac{v_p \cdot \Delta z}{2 \cdot v_p \cdot \Delta z} = \frac{1}{2}
\]
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- Making use of this in the above two equations,
- we obtain the following equations

\[
E_x^r(k, n + \frac{1}{2}) = E_x^r(k, n - \frac{1}{2}) + \frac{1}{2} \left[ H_y(k - \frac{1}{2}, n) - H_y(k + \frac{1}{2}, n) \right]
\]

\[
H_y(k + \frac{1}{2}, n + 1) = H_y(k + \frac{1}{2}, n) + \frac{1}{2} \left[ E_x^r(k, n + \frac{1}{2}) - E_x^r(k + 1, n + \frac{1}{2}) \right]
\]

- Hence the computer update equations (for free space \( \varepsilon_r(k) = 1 \)) are

\[
ex[k] = ex[k] + (0.5 / \varepsilon_r(k)) \times (hy[k-1] - hy[k])
\]

\[
hy[k] = hy[k] + 0.5 \times (ex[k] - ex[k+1])
\]
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- $k+1/2$ and $k-1/2$ are replaced by $k$ or $k-1$
- Note that the $n$ or $n+1/2$ or $n-1/2$ in the superscripts do not appear

- **FDTD Simulation of Gaussian pulse propagation in free space** (fdtd_1d_1.m)
- **FDTD Simulation of Gaussian pulse hitting a dielectric medium** (fdtd1_dielectric.m)
- From theoretical analysis of EM wave hitting a dielectric surface

\[
\tau = \frac{2\eta_2}{\eta_2 + \eta_1} = \frac{2\sqrt{\varepsilon_1}}{\sqrt{\varepsilon_1} + \sqrt{\varepsilon_2}}, \quad \Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{\sqrt{\varepsilon_1} - \sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1} + \sqrt{\varepsilon_2}}
\]
Fig. EM wave hitting a dielectric surface
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- For $\varepsilon_1=1.0$ and $\varepsilon_2=4.0$, we have $\Gamma = -\frac{1}{3}; \tau = \frac{2}{3}$
- FDTD simulation of Absorbing Boundary Condition
  - (fdtd_1d_ABC_boundary.m)
  - (fdtd_1d_no_boundary.m)
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- For $\varepsilon_1=1.0$ and $\varepsilon_2=4.0$, we have $\Gamma = -\frac{1}{3}; \tau = \frac{2}{3}$

- *FDTD simulation of Absorbing Boundary Condition* (fdtd_1d_ABC_boundary.m)

- (fdtd_1d_no_boundary.m)

- In calculating the E field,
  - we need to know the surrounding H values;
  - this is the fundamental assumption in FDTD

- At the edge of the problem space,
  - we will not have the value to one side,
  - but we know there are no sources outside the problem space
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- The wave moves $\Delta z/2 (= c_0 \cdot \Delta t)$ distance in one time step,
  - so it takes two time steps for a wave front to cross one cell
- Suppose we are looking for a boundary condition at the end where $k=1$
- Now if we write the $E$ field at $k=1$ as
  - $E_x (1,n) = E_x (2, n-2)$,
  - then the fields at the edge will not reflect
- This condition must be applied at both ends
2. FDTD Solution to Maxwell’s equations in 2-D Space

• In deriving 2-D FDTD formulation, we choose between one of two groups of three vectors each:
  • (a) Transverse magnetic (TM\(_z\)) mode,
    • which is composed of Ez, Hx, and Hy or
  • (b) Transverse electric (TE\(_z\)) mode,
    • which is composed of Ex, Ey, and Hz
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- Unlike time-harmonic guided waves,
  - none of the fields vary with z so that
  - there is no propagation in the z-direction
- But in general propagation along x- or y- directions or both is possible
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- *FDTD simulation of TM mode wave propagation* (fdtd_2d_TM_1.m)
- Expanding the Maxwell’s curl equations with
  - \( E_x = 0, E_y = 0, H_z = 0 \) and \( \partial / \partial z = 0 \),
- we obtain,
- 3 equations from the 6 equations
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- 3 equations as follows

\[
\frac{\partial E_z}{\partial t} = \frac{1}{\varepsilon} \left( \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \right)
\]

\[
\frac{\partial H_x}{\partial t} = \frac{1}{\mu} \left( \frac{\partial E_z}{\partial y} \right)
\]

\[
\frac{\partial H_y}{\partial t} = \frac{1}{\mu} \left( -\frac{\partial E_z}{\partial x} \right)
\]
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\[ H_x (i,j+1/2) \]
\[ H_y (i-1/2,j) \]
\[ E_z (i,j) \]
\[ H_x (i,j-1/2) \]
\[ H_y (i+1/2,j) \]
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- Observations:
  - Electric fields are calculated at integer space steps
  - Electric fields are calculated at half integer time steps
  - Magnetic fields are calculated at half integer space step and integer space step
  - Magnetic fields are calculated at integer time steps
- For example,
  - $H_y(i-1/2,j)$, $H_x(i,j-1/2)$, $H_y(i+1/2,j)$, $H_x(i,j+1/2)$
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- Central finite difference approximations of the 1st equation is as follows

\[
\frac{\partial E_z}{\partial t} = \frac{1}{\varepsilon} \left( \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \right)
\]

\[
E_z(i, j, n+\frac{1}{2}) - E_z(i, j, n-\frac{1}{2})
\]

\[
= \frac{\Delta t}{\varepsilon_0 \varepsilon_r}
\begin{bmatrix}
    H_y(i+\frac{1}{2}, j, n) - H_y(i-\frac{1}{2}, j, n) & H_x(i, j+\frac{1}{2}, n) - H_x(i, j-\frac{1}{2}, n)
\end{bmatrix}
\]

\[
\begin{bmatrix}
    \Delta x & \Delta y
\end{bmatrix}
\]
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- Central finite difference approximations of the next equation is as follows:

\[
\frac{\partial H_x}{\partial t} = \frac{1}{\mu} \left( \frac{\partial E_z}{\partial y} \right)
\]

\[
\frac{H_x(i, j + \frac{1}{2}, n+1) - H_x(i, j + \frac{1}{2}, n)}{\Delta t} = -\frac{1}{\mu_0 \mu_r} \frac{E_z(i, j+1, n+\frac{1}{2}) - E_z(i, j, n+\frac{1}{2})}{\Delta y}
\]
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- Central finite difference approximations of the next equation is as follows

\[
\frac{\partial H_y}{\partial t} = \frac{1}{\mu} \left( -\frac{\partial E_z}{\partial x} \right)
\]

\[
\frac{H_y(i + \frac{1}{2}, j, n+1) - H_x(i + \frac{1}{2}, j, n)}{\Delta t} = \frac{1}{\mu_0 \mu_r} \frac{E_z(i+1, j, n+\frac{1}{2}) - E_z(i, j, n+\frac{1}{2})}{\Delta x}
\]
Rearranging the above equations, we can write the final 3 update equations’ expressions as

Electric field update equation

\[
E_z\left(i, j, n+\frac{1}{2}\right) = E_z\left(i, j, n-\frac{1}{2}\right) + \frac{\Delta t}{\varepsilon_0\varepsilon_r} \left[ \frac{H_y\left(i+\frac{1}{2}, j, n\right) - H_y\left(i-\frac{1}{2}, j, n\right)}{\Delta x} - \frac{H_x\left(i, j+\frac{1}{2}, n\right) - H_x\left(i, j-\frac{1}{2}, n\right)}{\Delta y} \right]
\]
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- Magnetic field update equation

\[
H_x(i, j + \frac{1}{2}, n + 1) = H_x(i, j + \frac{1}{2}, n) - \frac{\Delta t}{\mu_0 \mu_r} \left( E_z(i, j + 1, n + \frac{1}{2}) - E_z(i, j, n + \frac{1}{2}) \right) / \Delta y
\]

\[
H_y(i + \frac{1}{2}, j, n + 1) = H_x(i + \frac{1}{2}, j, n) + \frac{\Delta t}{\mu_0 \mu_r} \left( E_z(i + 1, j, n + \frac{1}{2}) - E_z(i, j, n + \frac{1}{2}) \right) / \Delta x
\]

Electric field computer update equation is (i+1/2 and i-1/2 are replaced by i or i-1)

\[ E_z(i, j, n + \frac{1}{2}) = E_z(i, j, n - \frac{1}{2}) + \frac{\Delta t}{\varepsilon_0 \varepsilon_r} \left[ \frac{H_y(i + \frac{1}{2}, j, n) - H_y(i - \frac{1}{2}, j, n)}{\Delta x} - \frac{H_x(i, j + \frac{1}{2}, n) - H_x(i, j - \frac{1}{2}, n)}{\Delta y} \right] \]

- Electric field computer update equation is (i+1/2 and i-1/2 are replaced by i or i-1)
- \[ ez(i, j) = ez(i, j) + \{ dt / (eps_0*eps_r) \} \times [ \{ Hy(i, j) - Hy(i-1, j) \} / dx - \{ Hx(i, j) - Hx(i, j-1) \} / dy ] \]
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\[ H_x(i, j + \frac{1}{2}, n + 1) = H_x(i, j + \frac{1}{2}, n) - \frac{\Delta t}{\mu_0 \mu_r} \frac{E_z(i, j + 1, n + \frac{1}{2}) - E_z(i, j, n + \frac{1}{2})}{\Delta y} \]

- Magnetic field computer update equation is \((j+1/2\) is replaced by \(j\), usually \(\mu_r = 1\))
- \(H_x(i,j) = H_x(i,j) - \{dt/(\mu_0*dy)\} \ast \{Ez(i,j+1)-Ez(i,j)\}\)
Magnetic field computer update equation is (i+1/2 is replaced by i, usually \( \mu_r = 1 \))

\[
H_y\left( i + \frac{1}{2}, j, n + 1 \right) = H_x\left( i + \frac{1}{2}, j, n \right) + \frac{\Delta t}{\mu_0 \mu_r} \frac{E_z\left( i + 1, j, n + \frac{1}{2} \right) - E_z\left( i, j, n + \frac{1}{2} \right)}{\Delta x}
\]

- Magnetic field computer update equation is \((i+1/2 \text{ is replaced by } i, \text{ usually } \mu_r = 1)\)
- \(H_y(i, j) = H_y(i, j) + \left\{ \frac{\Delta t}{\mu_0 \mu_r \Delta x} \right\} \left\{ E_z(i+1, j) - E_z(i, j) \right\} \)
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- Some new MATLAB commands:
  - `image`: `IMAGE(C)` displays matrix C as an image
  - `imagesc`: data is scaled to use the full color map
  - `Colorbar`: appends a vertical color scale to the current axis
  - `Colormap`: a matrix may have any number of rows but it must have exactly 3 columns
  - Each row is interpreted as a color, with the first element specifying the intensity of red, the second green and the third blue
  - Color intensity is specified on the interval of 0.0 to 1.0
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- For example, [0 0 0] is black
- [1 1 1] is white
- [1 0 0] is red
- [0.5 0.5 0.5] is gray
- [127/255 1 212/255] is aquamarine
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*FDTD simulation of TM wave hitting a dielectric surface (right side), source at the center (2-D) (fdtd_2d_TM_dielectric.m)*

- Modify the above program as follows:
- Specify the position of the dielectric surface (id=..,jd=..)
- Choose a value of $\varepsilon_r=10.0..$
- Modify the constant value C1 accordingly
- Run the program
- In free space wave is propagating freely and
- on the other side reflection and transmission of the wave at the interface
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- Wave propagates more slowly on the medium on the right and
  - the pulse length is shorter

- FDTD simulation of TM wave propagation due to multiple sources (fdtd_2d_TM_multi_source.m)

3. 3-D FDTD

- From two Maxwell’s curl equations,
  - we need to consider all 6 equations for 3-D case

\[
\frac{\partial \vec{E}}{\partial t} = \frac{1}{\varepsilon_0 \varepsilon_r} \nabla \times \vec{H}; \quad \frac{\partial \vec{H}}{\partial t} = -\frac{1}{\mu_0 \mu_r} \nabla \times \vec{E}
\]
Fig. 4 Yee’s FDTD grid
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Yee’s FDTD grid

- Unlike 1-D and 2-D FDTD cases,
  - Electric fields are calculated at “integer” time-steps
    - and magnetic field at “half-integer” time-steps
  - Electric field components are placed at mid-points of the corresponding edges
- E.g.
  - $E_x$ is placed at midpoints of edges oriented along x-direction
  - $E_x$ is on half-grid in x and the integer grids in y & z
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- The magnetic field components are
  - placed at the centers of the faces of the cubes and
  - oriented normal to the faces
- E.g.
- Hx components are placed at the centers of the faces in the yz-plane
- Hence Hx is
  - on the integer grid in x and
  - on the half-grid in y & z
Electric fields are calculated at “integer” time-steps and magnetic field at “half-integer” time-steps

\[
\frac{\partial E_x}{\partial t} = \frac{1}{\varepsilon_r \varepsilon_0} \left( \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} \right)
\]

Ex is placed at midpoints of edges oriented along x-direction

\[
\frac{E_x^{n+1}(p+1/2, q, r) - E_x^n(p+1/2, q, r)}{\Delta t} =
\]

[\begin{align*}
&\frac{1}{\varepsilon_r \varepsilon_0} \left[ \frac{H_z^{n+1/2}(p+1/2, q+1/2, r) - H_z^{n+1/2}(p+1/2, q-1/2, r)}{\Delta y} \right] \\
&- \frac{1}{\varepsilon_r \varepsilon_0} \left[ \frac{H_y^{n+1/2}(p+1/2, q, r+1/2) - H_y^{n+1/2}(p+1/2, q, r-1/2)}{\Delta z} \right]
\end{align*}]

Hz is placed at center of faces in xy-plane

Hy is placed at center of faces in xz-plane
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\[
\frac{\partial E_y}{\partial t} = \frac{1}{\varepsilon_r \varepsilon_0} \left( \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} \right)
\]

\[
\frac{E_y^{n+1}(p, q+1/2, r) - E_y^n(p, q+1/2, r)}{\Delta t} = \frac{1}{\varepsilon_r \varepsilon_0} \left[ \frac{H_x^{n+1/2}(p, q+1/2, r+1/2) - H_x^{n+1/2}(p, q+1/2, r-1/2)}{\Delta z} \right]
\]

\[- \frac{1}{\varepsilon_r \varepsilon_0} \left[ \frac{H_z^{n+1/2}(p+1/2, q+1/2, r) - H_z^{n+1/2}(p-1/2, q+1/2, r)}{\Delta z} \right] \]
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\[ \frac{\partial E_z}{\partial t} = \frac{1}{\varepsilon_r \varepsilon_0} \left( \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \right) \]

\[ E_{z}^{n+1}(p, q, r+1/2) - E_{z}^{n}(p, q, r+1/2) = \frac{\Delta t}{\Delta t} \]

\[ \frac{1}{\varepsilon_r \varepsilon_0} \left[ \frac{H_y^{n+1/2}(p+1/2, q, r+1/2) - H_y^{n+1/2}(p-1/2, q, r+1/2)}{\Delta x} \right] \]

\[ - \frac{1}{\varepsilon_r \varepsilon_0} \left[ \frac{H_x^{n+1/2}(p, q+1/2, r+1/2) - H_x^{n+1/2}(p, q-1/2, r+1/2)}{\Delta y} \right] \]
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\[
\frac{\partial H_x}{\partial t} = -\frac{1}{\mu_0} \left( \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} \right)
\]

\[
\frac{H_x^{n+1/2}(p, q + 1/2, r + 1/2) - H_x^{n-1/2}(p, q + 1/2, r + 1/2)}{\Delta t} = \frac{1}{\mu_0} \left[ \frac{E_y^n(p, q + 1/2, r + 1) - E_y^n(p, q + 1/2, r)}{\Delta z} \right]
\]

\[
- \frac{1}{\mu_0} \left[ \frac{E_z^n(p, q + 1, r + 1/2) - E_z^n(p, q, r + 1/2)}{\Delta y} \right]
\]
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\[ \frac{\partial H_y}{\partial t} = - \frac{1}{\mu_0} \left( \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} \right) \]

\[ H_y^{n+1/2}(p+1/2, q, r+1/2) - H_y^{n-1/2}(p+1/2, q, r+1/2) = \frac{\Delta t}{\Delta x} \left( \frac{1}{\mu_0} \left[ E_z^n(p+1, q, r+1/2) - E_z^n(p, q, r+1/2) \right] \right) \]

\[ - \frac{1}{\mu_0} \left[ \frac{E_x^n(p+1/2, q, r+1) - E_x^n(p+1/2, q, r)}{\Delta z} \right] \]
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\[ \frac{\partial H_z}{\partial t} = - \frac{1}{\mu_0} \left( \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right) \]

\[ \frac{H_{z}^{n+1/2}(p+1/2, q+1/2, r) - H_{z}^{n-1/2}(p+1/2, q+1/2, r)}{\Delta t} = \]

\[ \frac{1}{\mu_0} \left[ \frac{E_x^n(p+1/2, q+1/2, r) - E_x^n(p+1/2, q, r)}{\Delta y} \right] \]

\[ - \frac{1}{\mu_0} \left[ \frac{E_y^n(p+1, q+1/2, r) - E_y^n(p, q+1/2, r)}{\Delta x} \right] \]
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- *FDTD simulation of cubical cavity (fdtd_3D_demo.m)*
- We will use FDTD to find
  - the resonant frequencies of an air-filled cubical cavity with metal walls
- Peaks in the frequency response indicate
  - the presence of resonant modes in the cavity
- An initial Hz field excites only
  - the TE modes
- The first four eigenfrequencies
  - \( TE_{101}, TE_{011}/TE_{201}, TE_{111} \) and \( TE_{102} \) modes
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- Field components placed on a grid with $3 \times 4 \times 2$ cells

- For $E_x$ ($3 \times 5 \times 3$) and $H_x$ ($4 \times 4 \times 2$)
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- Field components placed on a grid with $3\times4\times2$ cells

- For $E_y$ ($4\times4\times3$) and $H_y$ ($3\times5\times2$)
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- Field components placed on a grid with $3 \times 4 \times 2$ cells

- For $E_z (4 \times 5 \times 2)$ and $H_z (3 \times 4 \times 3)$
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- Artificial absorbers on the walls of anechoic chamber
Berenger’s PML is an artificial material that is theoretically designed to create no reflections regardless of the frequency, polarization and angle of incidence of a plane wave upon its interface.
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- PML could be analyzed in stretched coordinate systems.

- Source-free Maxwell’s equations

\[
\nabla_{\text{stretched}} \times \vec{E} = -j \omega \mu \vec{H}; \nabla_{\text{stretched}} \times \vec{H} = j \omega \varepsilon \vec{E};
\]

\[
\nabla_{\text{stretched}} \cdot (\varepsilon \vec{E}) = 0; \nabla_{\text{stretched}} \cdot (\mu \vec{H}) = 0;
\]

\[
\nabla_{\text{stretched}} = \hat{x} \frac{1}{s_x(x)} \frac{\partial}{\partial x} + \hat{y} \frac{1}{s_y(y)} \frac{\partial}{\partial y} + \hat{z} \frac{1}{s_z(z)} \frac{\partial}{\partial z}
\]

- Consider plane waves

\[
\vec{E} = \vec{E}_0 e^{-j\vec{k} \cdot \vec{r}}; \quad \vec{H} = \vec{H}_0 e^{-j\vec{k} \cdot \vec{r}}
\]

\[
\vec{k} \cdot \vec{r} = k_x x + k_y y + k_z z
\]
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- Substituting these into Maxwell’s equations, we have,
  \[ \mathbf{k}_{\text{stretched}} \times \mathbf{E} = \omega \mu \mathbf{H}; \mathbf{k}_{\text{stretched}} \times \mathbf{H} = -\varepsilon \mathbf{E}; \]
  \[ \mathbf{k}_{\text{stretched}} \cdot (\varepsilon \mathbf{E}) = 0; \mathbf{k}_{\text{stretched}} \cdot (\mu \mathbf{H}) = 0; \]

- Solution to the above equation is given by
  \[ k_x = ks_x \sin \theta \cos \phi, k_y = ks_y \sin \theta \sin \phi, k_z = ks_z \cos \theta \]
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- If $s_x$ is a complex number with a negative imaginary part,
  - the wave will be attenuated in the x direction

$$e^{-jkx} = e^{-jks_x \sin \theta \cos \phi_x} = e^{-jk(s' - js'') \sin \theta \cos \phi_x}, \quad s' \geq 1, \; s'' > 0$$

- Wave impedance is independent of coordinate stretching

$$\eta = \frac{|\vec{E}|}{|\vec{H}|} = \frac{\omega \mu}{|\vec{k}_{\text{stretched}}|} = \frac{\omega \mu}{\sqrt{k}} = \sqrt{\frac{\mu}{\varepsilon}}$$
Fig. 5 Computational domain truncated using the conductor backed PMLs

\[ s_x = s_z = 1, \quad s_y = s' - js' \]

\[ s_x = s' - js' \]

\[ s_y = s_z = 1, \]

\[ s_x = s' - js' \]
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References