A Saturation Algorithm for Homogeneous Binomial Ideals

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Summary

Let I be an ideal in the polynomial ring $k[\mathbf{x}]$ over a field k. Saturation of I by the product $x_1 \cdots x_n$, denoted by $I : (x_1 \cdots x_n)^\infty$ is the ideal $\{f \;:\; x_1^{a_1} \cdots x_n^{a_n} f \in I,\ a_i \geq 0, 1 \leq i \leq n\}$. Binomials in the ring are defined as polynomials with at most two terms [1]. Ideals with a binomial basis are called binomial ideals. Toric ideals are examples of homogeneous binomial ideals. We describe a fast algorithm to compute the saturation, $I : (x_1 \cdots x_n)^\infty$, of a homogeneous binomial ideal I. Here we would like to note that there are several algorithms to saturate pure difference binomial ideals, which are a special case of homogeneous binomial ideals, like the Sturmfels' saturation algorithm [3] and the Project and Lift algorithm [2].

Motivation

The primary motivation for the new approach is that the time complexity of Gröbner basis is a strong function of the number of variables. In the proposed algorithm, a Gröbner basis is computed in the i-th iteration in i variables. This requires the computation of a Gröbner basis over the ring $k[\mathbf{x}][U_i^{-1}]$. The Gröbner basis over such a ring is not known in the literature. Thus, we propose a variant of Gröbner bases, called pseudo Gröbner bases, and appropriately modify the Buchberger's algorithm to compute it.

Notations

Before proceeding, we will need some notations. U_i will denote the multiplicatively closed set $\{x_1^{a_1} \cdots x_{i-1}^{a_{i-1}} \;:\; a_j \geq 0, 1 \leq j < i\}$. $<_i$ will denote a graded reverse lexicographic term order with x_i being the smallest. $\varphi_i : k[\mathbf{x}] \to k[\mathbf{x}][U_{i-1}^{-1}]$ is the natural localization map $r \mapsto r/1$. A binomial of the form $cx^\alpha + dx^\beta$ is said to be *balanced*.

Our Approach

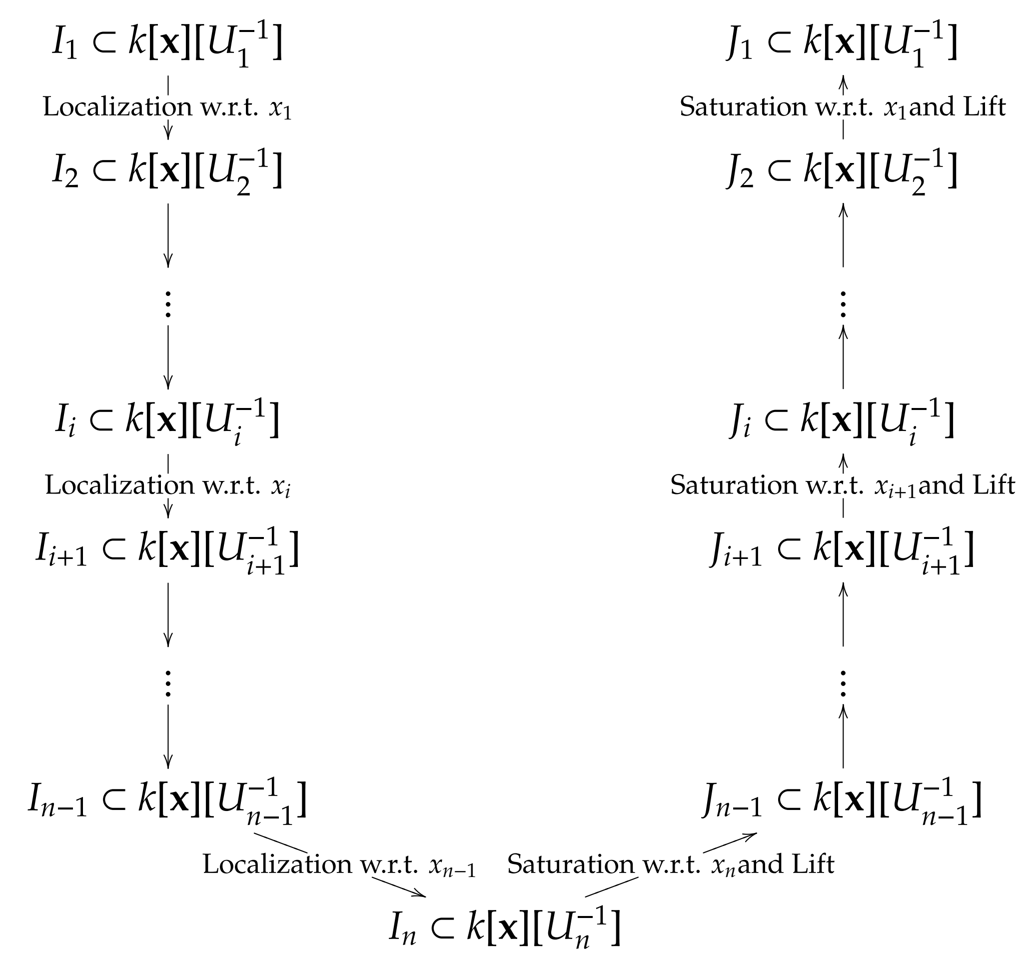
We propose a variant of Gröbner bases, called pseudo Gröbner bases, and appropriately modify the Buchberger's algorithm to compute it.

Definition A basis G of an homogeneous binomial ideal $I \subset k[\mathbf{x}][U_i^{-1}]$ is called a pseudo-Gröbner basis of I, if G can be partitioned into two sets G_1, G_2, such that -

1. G_2 contains balanced binomials only.
2. Every binomial of I reduces to 0 (mod G_2) by G_1, with respect to a given term-order.

Theorem Let (G_1, G_2) be a pseudo Gröbner basis of a homogeneous binomial ideal I in $k[\mathbf{x}][U_i^{-1}]$ with respect to $<_i$. Then $(G_1 : x_i^\infty, G_2 : x_i^\infty)$ is a pseudo Gröbner basis of $I : x_i^\infty$.

Our Approach



Algorithms

Data: A homogeneous binomial ideal, $I \subset k[\mathbf{x}]$.
Result: $I : (x_1, \ldots, x_n)^\infty$

for $i \leftarrow n$ ***to*** 1 **do**
  $G \leftarrow$ Gröbner basis of I w.r.t. $<_i$;
  $I \leftarrow \langle \{f \div (x_1, \ldots, x_n)^\infty | f \in G\} \rangle$;
end
return I ;

Algorithm 1: Sturmfels' Algorithm

Data: A homogeneous binomial ideal, $I \subset k[\mathbf{x}]$.
Result: $I : (x_1, \ldots, x_n)^\infty$

for $i \leftarrow n$ ***to*** 1 **do**
  $G \leftarrow$ Pseudo Gröbner basis of $\varphi_i(I)$ w.r.t. $<_i$;
  $I \leftarrow \langle \{\varphi_i^{-1}(f \div (x_1, \ldots, x_n)^\infty) | f \in G\} \rangle$;
end
return I ;

Algorithm 2: Proposed Algorithm

Preliminary Experimental Results

Number of variables	Size of basis: Initial	Size of basis: Final	Time taken (in sec.): Sturmfels'	Time taken (in sec.): Project and Lift	Time taken (in sec.): Proposed
8	4	186	0.30	0.12	0.10
	6	597	2.61	0.60	0.64
10	6	729	3.20	1.10	0.50
	8	357	2.40	0.40	0.29
12	6	423	1.70	0.90	0.27
	8	2695	305.00	60.00	27.20
14	10	1035	10.50	4.20	2.50

Table: Preliminary experimental results comparing Sturmfels', Project-and-Lift and our proposed algorithms

Conclusion

Our intuition as to why our algorithm is doing better, in these experiments, compared to Project and Lift is that their algorithm uses Sturmfels' saturation algorithm as a subroutine, though the extent to which it uses the algorithm depends on the input ideal. On the other hand, our algorithm computes all saturations by the same approach.

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