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Problem: [Arora, Barak - Exercise 2.15] In the VERTEX COVER problem, we are given an undirected graph G and an integer K and have to decide whether there is a subset S of at most K vertices such that every edge (i,j) of G, at least one of i or j is in S (such a subset is called a *vertex cover* of G). Prove that this problem is NP-complete.

Solution :

We know that the following problem is NP-complete :
P1: An independent set of a graph G = (V , E) is a VI ⊆ V such that no two vertices in VI share an edge. Does G have an independent set of size at least k ?

We will reduce it to following :
P2: A vertex cover of a graph G = (V , E) is a VC ⊆ V such that every (a, b) ∈ E is incident to at least a u ∈ VC . Does G have a vertex cover of size at most k ?

A : Vertex Cover is in N P.
Given VC , vertex cover of G = (V , E), |VC | <= k
We can check in O(|E | + |V |) that VC is a vertex cover for G . How?
For each vertex ∈ VC , remove all incident edges.
Check if all edges were removed from G .
Thus, Vertex Cover ∈ N P

B. Independent Set Problem can be reduced to Vertex Cover Problem in Polynomial Time.
Given a general instance of IS: G' =(V' , E'), k'
Construct a specific instance of VC: G =(V , E), k
V = V'
E = E'
(G = G')
k = |V'| − k'
This transformation is polynomial:
Constant time to construct G =(V , E)
O(|V |) time to count the number of vertices

Proof : G has an independent set VI of size k iff VC has a vertex cover VC of size at most k'.
Consider two sets I and J s.t. I ∩ J = ∅ and I ∪ J = V = V'

Given any edge (u, v), one of the following four cases holds:
u, v ∈ I
u ∈ I and v ∈ J
u ∈ J and v ∈ I
u, v ∈ J
Assume that I is an independent set of G then:
Case 1 cannot be : (vertices in I cannot be adjacent)
In cases 2 and 3, (u, v) has exactly one endpoint in J.
In case 4, (u, v) has both endpoints in J.
In cases 2, 3 and 4, (u, v) has at least one endpoint J.
Thus, vertices in J cover all edges of G . Also: |I| = |V| − |J| since I ∩ J = ∅ and I ∪ J = V = V
Thus, if I is an independent set of G , then J is a vertex cover of G (= G).
Similarly, we can prove that if J is a vertex cover for G , then I is an independent set for G.

Hence Proved.