Theory of Computation: Assignment 2
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Q1) Suppose $L_1, L_2 \in NP$. Then is $L_1 \cup L_2 \in NP$?

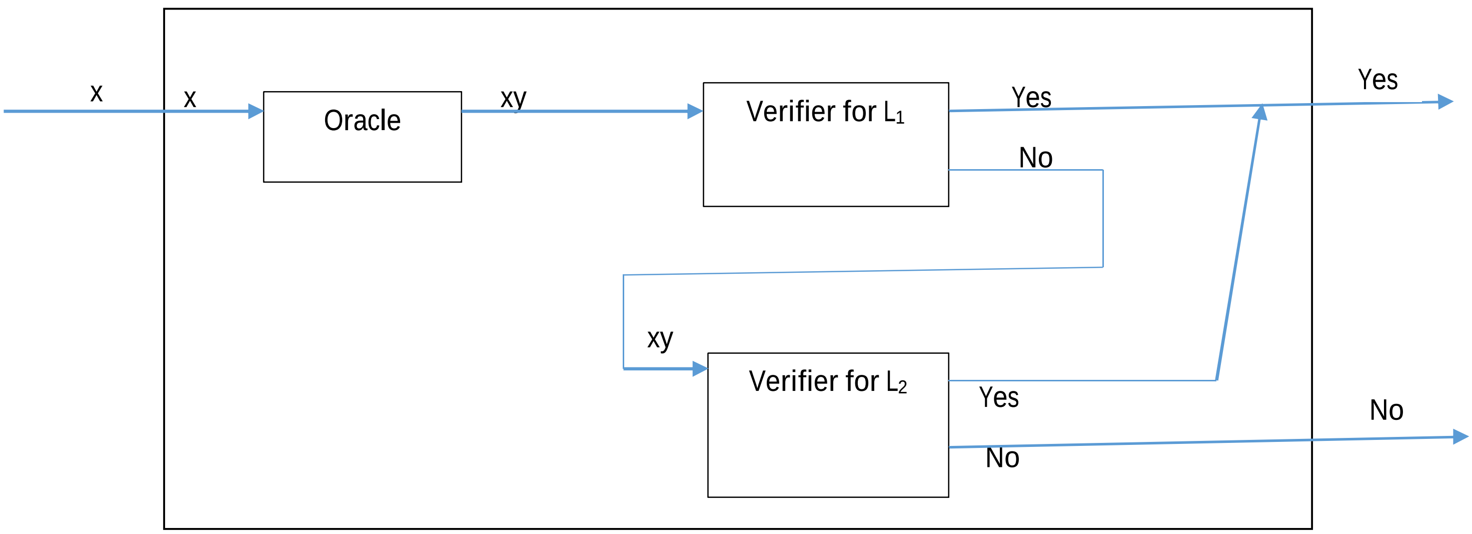
Ans: Yes, $L_1 \cup L_2 \in NP$. We will prove it as follows.

First of all, we note that a language L is in NP iff there exists a verification algorithm A (verifier A), and polynomials p, q such that:

- $L = L_A$;
- $\forall x \in L, \exists y$ such that $|y| \leq p(|x|)$ and A(x, y) = Yes;
- A on input (x, y) halts in time $\leq q(|x|)$.

[Note: We use (p + q)(n) to denote the polynomial p(n) + q(n)]

The diagram for our construction is as follows:



Oracle + Verifier for $L_1 U L_2$

Based on the above diagram:

Let $L_1, L_2 \in NP$, with verification algorithms A_1, A_2 (i.e., $L_1 = L_{A1}$, $L_2 = L_{A2}$), and polynomial bounds p_1, q_1, and p_2, q_2, respectively.

We know a define a new verification algorithm A as follows (it is based on the above figure):

```
A(x, y)
 if A1(x, y) = Yes
        then return Yes
else if  A2(x, y) = Yes
        then return Yes
else return No
```

We observe that A(x, y) = Yes iff A_1(x, y) = Yes or A_2(x, y) = Yes.

Now we will prove our construction:

1. To Prove: $L_1 \cup L_2 \subseteq L_A$.
   Proof: Let $x \in L_1 \cup L_2$. Then $x \in L_1$ or $x \in L_2$. If $x \in L_1$, then $\exists y$ such that $A_1(x, y)$ = Yes (as L_1 is an NP language). Hence, $A(x, y)$ = Yes. Similarly, if $x \in L_2$, then $\exists y$ such that $A_2(x, y)$ = Yes. Hence, $A(x, y)$ = Yes. Therefore $x \in L_A$.

2. To Prove: $L_A \subseteq L_1 \cup L_2$.
   Proof: Let $x \in L_A$. Then $\exists y$ such that $A(x, y)$ = Yes. This implies that $A_1(x, y)$ = Yes or $A_2(x, y)$ = Yes ,i.e., $x \in L_{A1}$ or $x \in L_{A2}$. Therefore $x \in L_{A1} \cup L_{A2} = L_1 \cup L_2$.

3. To prove: y is polynomially bounded by $|x|$.
   Proof: $\forall x \in LA$, $\exists y$ such that $A(x, y)$ = Yes. If $x \in L_1$, we have $|y| \le p_1(|x|)$. If $x \in L_2$, we have $|y| \le p_2(|x|)$. Therefore in either case, $|y| \le p_1(|x|) + p_2(|x|) = (p_1 + p_2)(|x|)$.

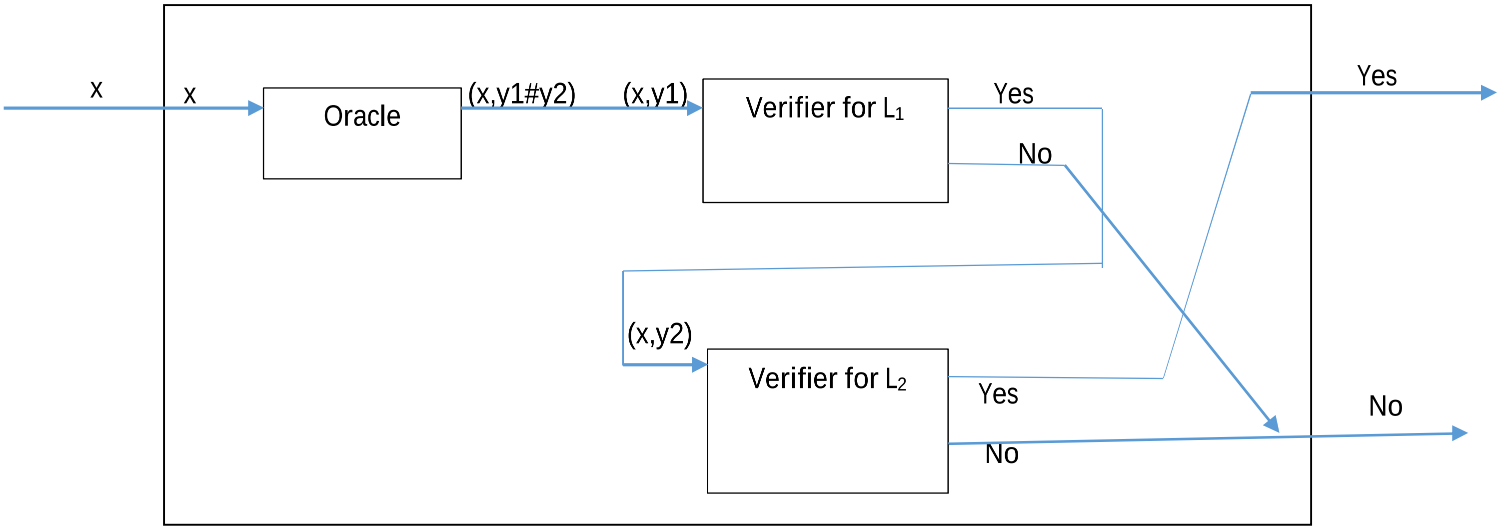
4. To prove: Time taken by verifier A is polynomially bounded by $|x|$.
   Proof: $\forall x \in L_A$, $\exists y$ such that $A(x, y)$ = Yes. If $x \in L_1$, then verifier A takes time $\le q_1(|x|)$. If $x \in L_2$, then verifier A takes time $\le q_1(|x|) + q_2(|x|)$. Therefore, in either case, time taken $\le q_1(|x|) + q_2(|x|) = (q_1 + q_2)(|x|)$.
   Thus A on (x, y) takes time $O((q_1 + q_2)(|x|))$ and is therefore polynomially bounded by $|x|$.

   Thus combining all the above results we have proved that $L_1 \cup L_2 \in NP$.

Q2) Suppose $L_1, L_2 \in NP$. Then is $L_1 \cap L_2 \in NP$?

Ans: Yes, $L_1 \cap L_2 \in NP$. We will prove it as follows.
Let $L_1, L_2 \in NP$, with verification algorithms A_1, A_2 and polynomial bounds p_1, q_1 and p_2, q_2, respectively. Moreover, let # be a distinguished character not in the alphabet of either L_1 or L_2. We now define a new verification algorithm A as follows:

```
 A(x, y)
 if y != y1#y2
    then return No
if A1(x, y1) = Yes
     then if A2(x, y2) = Yes
              then return Yes
return No
```



Oracle + Verifier for $L_1 \cap L_2$

Note that $A(x, y) = Yes$ iff $y = y1\#y2$ and $A_1(x, y1) = A_2(x, y2) = Yes$.

Now we will prove our construction.

1. To prove : $L_1 \cap L_2 \subseteq L_A$.

Proof: Let $x \in L_1 \cap L_2$. Then $x \in L_1$ and $x \in L_2$. Then, $\exists y1, y2$ such that $A_1(x, y1) = Yes$ and $A_2(x, y2) = Yes$. This implies that $A(x, y1\#y2) = Yes$. Therefore $x \in L_A$.

2. To prove: $L_A \subseteq L_1 \cap L_2$.

Proof: Let $x \in L_A$. Then $\exists y1\#y2$ such that $A(x, y1\#y2) = Yes$. This implies (by our construction) that $A_1(x, y1) = Yes$ and $A_2(x, y2) = Yes$. Hence $x \in L_{A1}$ and $x \in L_{A2}$. Therefore, $x \in L_1 \cap L_2$.

3. To prove: y is polynomially bounded by $|x|$.

Proof: $\forall x \in L_A$, $\exists y$ such that $A(x, y) = Yes$. Moreover, since $y = y1\#y2$, with $|y1| \leq p_1(|x|)$ and $|y2| \leq p_2(|x|)$, we have $|y| = |y1| + |y2| + 1 \leq (p_1 + p_2)(|x|) + 1$. Therefore $|y|$ is polynomially bounded by $|x|$.

4. A on (x, y) runs in time $O((q_1 + q_2)(|x| + |y|))$ [from our construction, it is true].

Thus combining all the above results we have proved that $L_1 \cap L_2 \in NP$.