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Problem: In the MAX CUT problem, we are given an undirected graph G and an integer K and have to decide whether there is a subset of vertices S such that there are at least K edges that have one endpoint in S and one endpoint in V-S where V is the set of vertices of graph G. Prove that this problem is NP-complete.

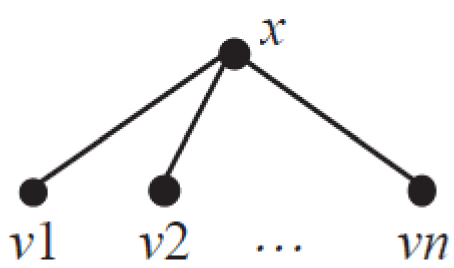
Solution:

Intuition:- We will reduce INDSET problem to the given MAX CUT problem.

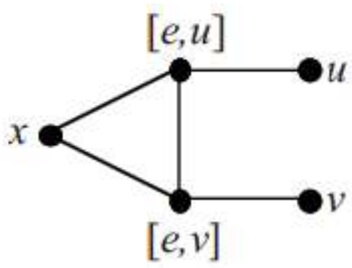
Proof:- Reduction from INDSET to MAX CUT is as follows:

V = {v1,v2,...,vn} (set of vertices for INDSET)

maps to



Each edge e = uv maps to a "gadget" that looks like



The MAX CUT possible for each gadget in different cases are as follows:

If u and v are both on the same side of the cut as x, put [e,u] and [e,v] on the opposite side (4 edges cross the cut). If only u is with x, put [e,u] on the opposite side (same with only v, in this case also 4 edges cross the cut). If neither u nor v is with x, at most 3 edges will cross the cut.

Where x is a new vertex in the modified graph and also [e,u] and [e,v] are also new vertices for each edge in the original graph G.

So, the transformations are as follows:

Let the instance of **INDSET** be described by $G = (V,E)$ and k. Then the **MAX-CUT** instance is $G' = (V',E')$ and k', where

$V' = V \cup \{x\} \cup V''$

$V'' = \{[e,u],[e,v] \mid e = uv \in E\}$

$E' = \{xv \mid v \quad V\} \cup E''$

$E'' = \{x[e,u], x[e,v], [e,u][e,v], [e,u]u, [e,v]v \mid e = uv \in E\}$

$K' = k + c.|E| \quad$ where $c = 4$

V'' is the set of extra gadget vertices; E'' the gadget edges.

Gadget property (for a gadget based on $e = uv$): If at least one of u or v is on the same side of the cut as x, the gadget vertices can be assigned so that c (= 4) gadget edges (but no more) cross the cut. Otherwise at most c-1 gadget edges cross the cut.

Proof that INDSET implies MAX CUT:

Let S be a INDSET with |S|≥ K in graph G. Now, in the graph G' take all the vertices of set S on one side of the cut and x on the opposite side of the cut so we have k edges crossing the cut till now. We also have |E| gadgets in G' and due to the property of INDSET if uv ϵ E then both of them together cannot belong to S at most one can belong to S. So, two cases are possible:

Case 1: u, v both don't belong to S then we can assign both of them on the side of x and can thus generate a MAX CUT of 4 from the gadget.

Case 2: One belong to S either u or v then as said above that one vertex will belong to opposite side of x and in this case also we can optimally manage our gadget to generate a MAX CUT of 4.

So, each gadget can generate 4 cross edges and we already have K cross edges and also there are |E| number of gadgets according to our construction.

Hence, $K' = K + 4.|E|$

Therefore, **INDSET** implies **MAX CUT**

Proof that MAX CUT implies INDSET:

- Let (S', V'-S') be the cut that satisfies the **MAX-CUT** instance and let E_c be the set of edges crossing the cut. We know that $|E_c| \geq K' = K + 4.|E|$.
- Each edge gadget has at most $c = 4$ edges in E_c implies at least k edges of E_c are not gadget edges.
- Non-gadget edges are of the form xv for some $v \in V$. Let S_c be $\{v \in V \mid xv \in E_c\}$ and let $|S_c| = K + l$.
- There are $\leq l$ edges of G whose gadgets have fewer than 4 cut edges. Call these edges E_v (for *violations*).
- For each $e \in E_v$ choose one endpoint and remove it from S_c - call the remaining vertices S.
- S is an independent set - no edge has more than one endpoint in S - and $|S| \geq K$ - $|S_c| \geq K + l$ and $\leq l$ vertices were removed to form S - so S is the is an INDSET with $|S| \geq K$.

Therefore, **MAX CUT** implies **INDSET**

Hence, Proved