CS-301

Assignment 1

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Problem: Prove that the language HAMCYCLE of undirected graphs that contain Hamltonian cycle (a simple cycle involving all the vertices) is NP-complete.
Solution: To prove that HAMCYCLE is NP-Complete we have to prove two things. First, that it is NP. And then that it is NP-Hard(i.e. every language in NP can be reduced to it).

Part 1

To prove that HAMCYCLE is NP we need to ensure two things:

- First that the length of the solution is polynomial in the input size. This is trivial as the certificate will consist of a ordered sequence of edges which form the Hamiltonian cycle which in turn will be a subset of the edge set, E.
- Next we need to show that the verifier machine will also run in polynomial time. This can be done simply as we just scan through the certificate and ensure that the end vertex of one edge is the start vertex of next edge. And finally we have to ensure that start vertex of first edge is the end vertex of last edge.

  Thus, we have shown that HAMCYLCE $\in$ NP.

Part 2

To show that HAMCYCLE is NP-Hard we'll follow the following series of reductions :

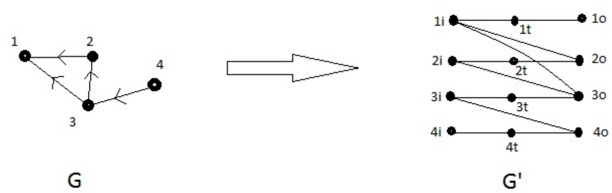
$$\text{3-SAT} \leq_p \text{dHAMPATH} \leq_p \text{HAMPATH} \leq_p \text{HAMCYCLE}$$

- 3-SAT $\leq_p$ dHAMPATH : This has already been done in the class.
- dHAMPATH $\leq_p$ HAMPATH : Given a directed graph G we need to convert it to an undirected graph G' s.t. a Hamiltonian path exists in G iff a Hamiltonian path exists in G'.
  First we provide the algorithm to construct G' from G.
  - For every vertex x in G there will be 3 vertices x_i, x_o and x_t.
  - There exists edges (x_i, x_t) and (x_t, x_o).

- For every edge in G of the form (x, y) there exists an edge (x_o, y_i) in G'.

If G contains m vertices and n edges then G will consist of $3 * m$ vertices and $2 * m + n$ edges which are clearly polynomial in the input size.

Example of this algorithm can be seen below:



Now, we prove that the above reduction works.

- If a Hamiltonian path exists in G then a Hamiltonian path exists in G'.
  Let the Hamiltonian path of G be of the form (x_1, x_2),(x_2, x_3) ... (x_{m-1}, x_m) then the Hamiltonian path in G' will be of the form (x_{1i}, x_{1t}),(x_{1t}, x_{1o}),(x_{1o}, x_{2i}),(x_{2i}, x_{2t}) ... $(x_{(m-1)o}, x_{(m)i})$,$(x_{(m)i}, x_{(m)t})$,$(x_{(m)t}, x_{(m)o})$.
- If a Hamiltonian path exists in G' then a Hamiltonian path exists in G.
  Let a Hamiltonian path exist in G'. Case 1: The starting and ending vertices of the hamiltonian path are not of the form x_t. In this case either the start or the end vertex will be of the form x_o. Starting from this vertex follow the hamiltonian path, the edges encountered will alternate between (x_o, y_i) and a pair of edges (y_i, y_t), (y_t, y_o). For every edge of the form (x_o, y_i) pick the edge starting from x and ending at y in the Graph G in the set of edges forming the hamiltonian path in G.
  Case 2: The starting vertex of the path is of the form x_t. Then one of second vertex and ending vertex will be of the form x_o. Start from such a vertex. The edges encountered will alternate between (x_o, y_i) and a pair of edges (y_i, y_t), (y_t, y_o). For every edge of the form (x_o, y_i) pick the edge starting from x and ending at y in the Graph G in the set of edges forming the hamiltonian path in G.

Thus, we have reduced dHAMPATH to HAMPATH.

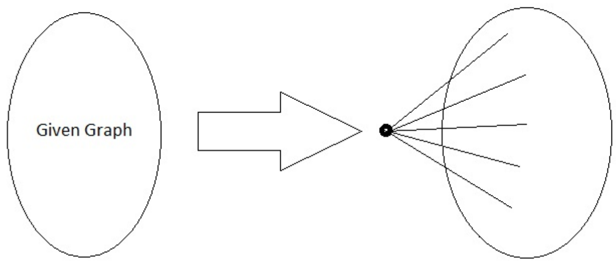
- HAMPATH $\leq_p$ HAMCYCLE : Given an undirected graph G we need to convert it to an undirected graph G' s.t. G contains a Hamiltonian path iff G' contains a Hamiltonian cycle.
  First we provide the algorithm to construct G' from G.

  - G' contains all the nodes of G and an additional node v.

- G' contains all the edges of G and additional edges of the form (x, v) for every vertex x of G.

If G contains m vertices and n edges then G will consist of $m + 1$ vertices and $2 * m + n$ edges which are clearly polynomial in the input size.

Example of this algorithm can be seen below:



Now, we prove that the above reduction works.

- If G contains a Hamiltonian path then G' consists of a Hamiltonian cycle.
  Let the Hamiltonian path of G be of the form $(x1, x2)$,$(x2, x3)$, ... ,(x_{m-1}, x_m) then the Hamiltonian cycle of G' will be of the form $(v, x1)$,$(x1, x2)$,$(x2, x3)$, ... ,(x_{m-1}, x_m),(x_m, v).
- If G' contains a Hamiltonian cycle then G consists of a Hamiltonian path.
  Let the Hamiltonian cycle of G' be of the form $(v, x1)$,$(x1, x2)$,$(x2, x3)$, ... ,(x_{m-1}, x_m),(x_m, v) then the Hamiltonian path of G will be of the form $(x1, x2)$,$(x2, x3)$, ... ,(x_{m-1}, x_m).

Thus, we have reduced HAMPATH to HAMCYCLE.